



Reduced Weyl asymptotics for pseudodifferential operators on bounded domains II.

The compact group case [☆]

Roch Cassanas, Pablo Ramacher

Georg-August-Universität Göttingen, Institut für Mathematik, Bunsenstr. 3-5, 37073 Göttingen, Germany

Received 4 October 2007; accepted 28 January 2008

Available online 10 April 2008

Communicated by Paul Malliavin

Abstract

Let $G \subset O(n)$ be a compact group of isometries acting on n -dimensional Euclidean space \mathbb{R}^n , and \mathbf{X} a bounded domain in \mathbb{R}^n which is transformed into itself under the action of G . Consider a symmetric, classical pseudodifferential operator A_0 in $L^2(\mathbb{R}^n)$ that commutes with the regular representation of G , and assume that it is elliptic on \mathbf{X} . We show that the spectrum of the Friedrichs extension A of the operator $\text{res} \circ A_0 \circ \text{ext}: C_c^\infty(\mathbf{X}) \rightarrow L^2(\mathbf{X})$ is discrete, and using the method of the stationary phase, we derive asymptotics for the number $N_\chi(\lambda)$ of eigenvalues of A equal or less than λ and with eigenfunctions in the χ -isotypic component of $L^2(\mathbf{X})$ as $\lambda \rightarrow \infty$, giving also an estimate for the remainder term for singular group actions. Since the considered critical set is a singular variety, we recur to partial desingularization in order to apply the stationary phase theorem.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Pseudodifferential operators; Asymptotic distribution of eigenvalues; Compact group actions; Peter–Weyl decomposition; Partial desingularization

Contents

1. Introduction	92
2. Reduced spectral asymptotics and the approximate spectral projection operators	95

[☆] This research was financed by the grant RA 1370/2-1 of the German Research Foundation (DFG).

E-mail addresses: cassanas@uni-math.gwdg.de (R. Cassanas), ramacher@uni-math.gwdg.de (P. Ramacher).

3. Compact group actions and the principle of the stationary phase	101
4. Phase analysis and partial desingularization	108
5. Computation of the leading term	115
6. Proof of the main result	126
References	127

1. Introduction

Let $G \subset O(n)$ be a compact Lie group of isometries acting on Euclidean space \mathbb{R}^n , and \mathbf{X} a bounded open set of \mathbb{R}^n which is transformed into itself under the action of G . Consider the regular representation of G

$$T(k)\varphi(x) = \varphi(k^{-1}x) \quad (1)$$

in the Hilbert spaces $L^2(\mathbb{R}^n)$ and $L^2(\mathbf{X})$ of square-integrable functions by unitary operators. As a consequence of the Peter–Weyl theorem, T decomposes into isotypic components according to

$$L^2(\mathbb{R}^n) = \bigoplus_{\chi \in \hat{G}} \mathcal{H}_\chi, \quad L^2(\mathbf{X}) = \bigoplus_{\chi \in \hat{G}} \text{res } \mathcal{H}_\chi,$$

where \hat{G} denotes the set of irreducible characters of G , and $\text{res}: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbf{X})$ is the natural restriction operator. The spaces \mathcal{H}_χ are closed subspaces, and the corresponding orthogonal projection operators are given by

$$P_\chi = d_\chi \int_G \overline{\chi(k)} T(k) dk, \quad (2)$$

where $d_\chi = \chi(\mathbf{1})$ is the dimension of any irreducible representation belonging to the character χ , and dk denotes the normalized Haar measure on G . In what follows, we do not assume that the boundary $\partial\mathbf{X}$ of \mathbf{X} is smooth, but only that there exists a constant $c > 0$ such that for any sufficiently small $\varrho > 0$, $\text{vol}(\partial\mathbf{X})_\varrho \leq c\varrho$, where $(\partial\mathbf{X})_\varrho = \{x \in \mathbb{R}^n: \text{dist}(x, \partial\mathbf{X}) < \varrho\}$, and that $0 \notin \partial\mathbf{X}$.

Consider now a symmetric, classical pseudodifferential operator A_0 in \mathbb{R}^n of order $2m$ that commutes with the operators $T(k)$ for all $k \in G$. Let a_{2m} be its principal symbol, and assume that there exists a constant $C_0 > 0$ such that

$$a_{2m}(x, \xi) \geq C_0 |\xi|^{2m}, \quad \forall x \in \mathbf{X}, \quad \forall \xi \in \mathbb{R}^n. \quad (3)$$

If we write $\text{ext}: C_c^\infty(\mathbf{X}) \rightarrow L^2(\mathbb{R}^n)$ for the natural extension operator by zero, it turns out that under condition (3), the operator

$$\text{res} \circ A_0 \circ \text{ext}: C_c^\infty(\mathbf{X}) \rightarrow L^2(\mathbf{X})$$

is symmetric and lower semi-bounded, and we denote its Friedrichs extension by A . It can be shown that A has compact resolvent, and if the boundary of \mathbf{X} is sufficiently smooth, and A_0 satisfies the transmission property, the domain of A is given by

$$D(A) = \{u \in H_0^m(\mathbf{X}): A_0 u \in L^2(\mathbf{X})\},$$

where $H_0^m(\mathbf{X})$ is the closure of $C_c^\infty(\mathbf{X})$ in the Sobolev space $H^m(\mathbf{X})$, so that we are in presence of a generalized Dirichlet problem. Since A leaves invariant each of the isotypic components $\text{res } \mathcal{H}_\chi$, the restriction of A to $\text{res } \mathcal{H}_\chi$ gives rise to the so-called reduced operator A_χ . Its domain is $D(A_\chi) = D(A) \cap \text{res } \mathcal{H}_\chi$, and its spectrum is discrete, the spectrum of A being equal to the union of the spectra of the operators A_χ .

The purpose of this paper is to investigate the spectral counting function $N_\chi(\lambda)$ of A_χ , which is given by the number of eigenvalues of A_χ , counting multiplicities, that are less than $\lambda \in \mathbb{R}$. It corresponds to the number of eigenvalues of A less than λ , and with eigenfunctions in the χ -isotypic component of $L^2(\mathbf{X})$, so that

$$N_\chi(\lambda) = d_\chi \sum_{t \leq \lambda} \mu_\chi(t),$$

where $\mu_\chi(t)$ denotes the multiplicity of any irreducible representation ϱ_χ of dimension d_χ corresponding to the character χ in the eigenspace of A with eigenvalue t . $N_\chi(\lambda)$ describes the distribution of eigenvalues of A , and we shall investigate its asymptotic behavior as $\lambda \rightarrow +\infty$ by means of the generalized theorem of the stationary phase. It will turn out that $N_\chi(\lambda)$ is intimately related to the representation theory of G , and the geometry of the Hamiltonian action of G on the symplectic manifold $T^*(\mathbf{X})$. In fact, if (A_1, \dots, A_d) is a basis of the Lie algebra \mathfrak{g} of G , let

$$\mathbb{J}: T^*(\mathbf{X}) \simeq \mathbf{X} \times \mathbb{R}^n \rightarrow \mathfrak{g} \simeq \mathbb{R}^d, \quad (x, \xi) \rightarrow (\langle A_1 x, \xi \rangle, \dots, \langle A_d x, \xi \rangle),$$

be the associated momentum map, where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product in \mathbb{R}^n , and denote by

$$\Omega_0/G = \mathbb{J}^{-1}(\{0\})/G$$

the symplectic quotient of $T^*(\mathbf{X})$ at level zero. This quotient is naturally related to the critical set of the phase function in question, and plays a crucial role in our reduction. Indeed, we shall prove that $N_\chi(\lambda)$ is asymptotically determined by a certain volume of the quotient Ω_0/G , which is symplectically diffeomorphic to $T^*(\mathbf{X}/G)$ on its smooth part [8]. Now, the major difficulty in applying the generalized stationary phase theorem in our setting stems from the fact that, due to the singular orbit structure of the underlying group action, the zero level Ω_0 of the momentum map, and, consequently, the considered critical set, are in general singular varieties. In fact, if the G -action on $T^*(\mathbf{X})$ is not free, the considered momentum map is no longer a submersion, so that Ω_0 and Ω_0/G are not smooth anymore. Nevertheless, it can be shown that these spaces have a Whitney stratification into smooth submanifolds, see [20, Theorems 8.3.1 and 8.3.2], which corresponds to the stratification of $T^*(\mathbf{X})$, and \mathbb{R}^n into orbit types. To apply the principle of the stationary phase to our problem, we shall therefore proceed to partially resolve the singularities of Ω_0 , and then apply the stationary phase theorem in the resolution space under the sole assumption that the set $\text{Sing } \Omega_0$ of points where Ω_0 is not a manifold is contained in a strict vector

subspace of $T^*(\mathbf{X})$. This is always fulfilled for group actions that satisfy the following condition¹: if H_0 is a closed subgroup of G , and if $\mathbb{R}_{(H_0)}^n$ denotes the union of all principal orbits in \mathbb{R}^n of type G/H_0 , which is an open and dense subset in \mathbb{R}^n , then $\mathbb{R}^n \setminus \mathbb{R}_{(H_0)}^n$ should be contained in a strict vector subspace of \mathbb{R}^n . The main result of this paper is Theorem 8, which states that, as $\lambda \rightarrow +\infty$, one has the asymptotic formula

$$N_\chi(\lambda) = \frac{d_\chi[\varrho_{\chi|H_0} : 1]}{(2\pi)^{n-\kappa}} \text{vol}([a_{2m}^{-1}((-\infty, 1]) \cap \Omega_0]/G) \lambda^{(n-\kappa)/2m} + O(\lambda^{(n-\kappa-1/4)/2m}),$$

where $d_\chi = \chi(\mathbf{1})$, $[\varrho_{\chi|H_0} : 1]$ is the multiplicity of the trivial representation in the restriction of ϱ_χ to any principal isotropy group conjugated to H_0 , and κ the common dimension of the orbits of principal type. The volume of the quotient $[a_{2m}^{-1}((-\infty, 1]) \cap \Omega_0]/G$ is defined in Section 5.

The asymptotic distribution of eigenvalues was first studied by Weyl [25] for certain second order differential operators in \mathbb{R}^n using variational techniques. Hörmander [13] then extended these results to elliptic pseudodifferential operators on closed manifolds using the theory of Fourier integral operators. The first ones to study reduced Weyl asymptotics for elliptic operators on closed Riemannian manifolds in the presence of a compact group of isometries were Donnelly [6] together with Brüning and Heintze [3]. In the semi-classical context, reduced Weyl asymptotics and trace formulae were investigated in [7], and in [4] via coherent states. Our approach is based on the method of approximate spectral projections, first introduced by Tulovskii and Shubin [24]. Nevertheless, due to the presence of the boundary, the original method cannot be applied to our situation, and one has to use more elaborate techniques, which were subsequently developed by Feigin [9] and Levendorskii [18]. Compared to the method of Fourier integral operators, this approach gives weaker estimates for the remainder, but allows to consider non-smooth boundaries. Recently, Bronstein and Ivrii have obtained even sharp estimates for the remainder term in the case of differential operators on manifolds with boundaries satisfying the conditions specified above [2,16].

This paper is the second part of an investigation initiated in [21], which we shall refer to in the following as Part I. There, the foundations of the calculus of approximate spectral projection operators were provided, and the case of a finite group of isometries was settled. In this second part, the case of a compact group of isometries will be considered. Before we start, some comments on the results obtained might be in place. Asymptotics for the spectral counting function $N_\chi(\lambda)$ were obtained in [6] and [3] for general compact, isometric and effective Lie group actions using heat kernel methods; nevertheless, this approach does not allow to derive estimates for the remainder term. Using Fourier integral operator techniques, the same authors obtained rather optimal remainder estimates for compact G -manifolds in the cases where there is only one orbit type, or all orbits have the same dimension. For orthogonal actions in \mathbb{R}^n , estimates for the remainder were obtained in [7,12] in case that the union $\mathbb{R}_{(H_0)}^n$ of all principal orbits is given by $\mathbb{R}^n \setminus \{0\}$. In this paper, remainder estimates are obtained for the first time for a large class of singular group actions by partially resolving the singularities of the zero level of the momentum map Ω_0 . We remark that our method could be easily adapted to the setting of Fourier integral operators as e.g. described in [12] or [7], extending previous results to more general situations, which would even yield optimal remainder estimates.

¹ Examples for such group actions are given in Remark 1.

2. Reduced spectral asymptotics and the approximate spectral projection operators

In this section, we shall review some basic facts in the theory of pseudodifferential operators that will be needed in the sequel, and introduce the method of approximate spectral projection operators. For a more detailed exposition, the reader is referred to Part I, Sections 2 and 3. Let A_0 be a classical pseudodifferential operator of order $2m$ in \mathbb{R}^n , regarded as an operator in $L^2(\mathbb{R}^n)$ with domain $C_c^\infty(\mathbb{R}^n)$. In other words, A_0 can be represented by an oscillatory integral of the form

$$A_0 u(x) = \iint e^{i(x-y)\xi} a(x, \xi) u(y) dy d\xi,$$

where its symbol $a(x, \xi)$ has an asymptotic expansion of the form

$$a(x, \xi) \sim \sum_{j \geq 0} a_{2m-j}(x, \xi) (1 - \chi(\xi)),$$

χ being a compactly supported function equal to 1 in a neighborhood of zero, and the functions a_{2m-j} are homogeneous of degree $2m - j$ in variable ξ . a_{2m} is called the principal symbol of A_0 . If $0 \leq \varrho, \delta \leq 1$, and \mathbf{Y} is an open set in \mathbb{R}^n , let us denote by $S_{\varrho, \delta}^m(\mathbf{Y} \times \mathbb{R}^n)$ the set of smooth functions $\sigma(x, \xi)$ on $\mathbf{Y} \times \mathbb{R}^n$ such that for all compact sets K in \mathbf{Y} , and all multi-indices α, β , there exist constants $C_{K, \alpha, \beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)| \leq C_{K, \alpha, \beta} \langle \xi \rangle^{m - \varrho|\alpha| + \delta|\beta|}.$$

Let $L_{\varrho, \delta}^m(\mathbf{Y})$ be the class of pseudodifferential operators with symbols in $S_{\varrho, \delta}^m(\mathbf{Y} \times \mathbb{R}^n)$. Then, as a local pseudodifferential operator, $A_0 \in L_{1,0}^{2m}(\mathbb{R}^n)$, see [23, Section 3.7]. In what follows, we shall also need certain global spaces of symbols and pseudodifferential operators, which also take decay properties in x into account. They were introduced by Hörmander within the framework of Weyl calculus of pseudodifferential operators. Thus, consider on \mathbb{R}^{2n} the metric

$$\tilde{g}_{x, \xi}(y, \eta) = (1 + |x|^2 + |\xi|^2)^\delta |y|^2 + (1 + |x|^2 + |\xi|^2)^{-\varrho} |\eta|^2,$$

where $1 \geq \varrho > \delta \geq 0$, and put $h(x, \xi) = (1 + |x|^2 + |\xi|^2)^{-1/2}$.

Definition 1. Let p be a \tilde{g} -continuous function. The class $\Gamma_{\varrho, \delta}(p, \mathbb{R}^{2n})$, $0 \leq \delta < \varrho \leq 1$, consists of all functions $u \in C^\infty(\mathbb{R}^{2n})$ which for all multiindices α, β satisfy the estimates

$$|\partial_\xi^\alpha \partial_x^\beta u(x, \xi)| \leq C_{\alpha\beta} p(x, \xi) (1 + |x|^2 + |\xi|^2)^{(-\varrho|\alpha| + \delta|\beta|)/2}.$$

In particular, we shall write $\Gamma_{\varrho, \delta}^l(\mathbb{R}^{2n})$ for $\Gamma_{\varrho, \delta}(h^{-l}, \mathbb{R}^{2n})$, where $l \in \mathbb{R}$.

The class $\Gamma_{\varrho, \delta}(p, \mathbb{R}^{2n})$ is also denoted by $S(\tilde{g}, p)$, see Part I, Definitions 1 and 3. Let now $a \in \Gamma_{\varrho, \delta}(p, \mathbb{R}^{2n})$, $0 \leq 1 - \varrho \leq \delta < \varrho \leq 1$, and $\tau \in \mathbb{R}$. Then

$$Au(x) = \iint e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy d\xi$$

defines a continuous operator in $\mathcal{S}(\mathbb{R}^n)$, respectively $\mathcal{S}'(\mathbb{R}^n)$, see Part I, Corollary 1. In this case, a is called the τ -symbol of A , and the operator A is denoted by $\text{Op}^\tau(a)$. If $\tau = 1/2$, a is called the Weyl symbol of A , and one also writes $\text{Op}^w(a)$ for A . Pseudodifferential operators with real Weyl symbols give rise to self-adjoint operators. For $\tau = 0$ and $\tau = 1$ one simply obtains the usual left and right symbols, respectively. Our symbol classes will be mainly of the form $S(h^{-2\delta}g, p) = \Gamma_{1-\delta, \delta}(p, \mathbb{R}^{2n})$ with

$$g_{x, \xi}(y, \eta) = |y|^2 + h(x, \xi)^2 |\eta|^2,$$

where p is a smooth, positive, g -continuous function, and $0 \leq \delta < 1/2$. In what follows, $\Pi_{\varrho, \delta}(p, \mathbb{R}^n)$ and $\Pi_{\varrho, \delta}^l(\mathbb{R}^n)$ will denote the classes of pseudodifferential operators with symbols in $\Gamma_{\varrho, \delta}(p, \mathbb{R}^{2n})$, and $\Gamma_{\varrho, \delta}^l(\mathbb{R}^{2n})$, respectively.

Consider now a bounded domain \mathbf{X} in \mathbb{R}^n with not necessarily smooth boundary $\partial\mathbf{X}$, and let a be the left symbol of the classical pseudodifferential operator A_0 . Clearly, $a \in S(g, h^{-2m}, Z \times \mathbb{R}^n)$ for any compact set $Z \subset \mathbb{R}^n$. By changing a outside $\mathbf{X} \times \mathbb{R}^n$, we can therefore assume that $a \in S(g, h^{-2m})$, so that

$$A_0 \in \Pi_{1,0}^{2m}(\mathbb{R}^n).$$

Assume now that A_0 satisfies the ellipticity condition (3).

Lemma 1. *The ellipticity condition (3) is equivalent to the existence of constants $C, M > 0$ such that*

$$\left((A_0 + M\mathbf{1})u, u \right)_{L^2(\mathbf{X})} \geq C \|u\|_{H^m(\mathbf{X})}^2, \quad \forall u \in C_c^\infty(\mathbf{X}), \quad (4)$$

where $\|\cdot\|_{H^m(\mathbf{X})}$ is the norm in the Sobolev space $H^m(\mathbf{X})$.

Proof. Since $A_0 + M\mathbf{1}$ is a classical symmetric pseudodifferential operator with principal symbol a_{2m} , the implication (4) \Rightarrow (3) follows with [18, Lemma 13.1]. Now, let us assume that (1) is fulfilled. By compactness, there exists a constant $\varepsilon > 0$ such that, if $\mathbf{X}_\varepsilon = \{x \in \mathbb{R}^n : \text{dist}(x, \mathbf{X}) < \varepsilon\}$, one has

$$a_{2m}(x, \xi) \geq \frac{C_0}{2} |\xi|^{2m}, \quad \forall x \in \mathbf{X}_\varepsilon, \quad \forall \xi \in \mathbb{R}^n. \quad (5)$$

The restriction of A_0 to \mathbf{X}_ε is of course in $L_{1,0}^{2m}(\mathbf{X}_\varepsilon)$ since \mathbf{X} is bounded, and is elliptic in view of (5). It is not properly supported in general but, according to [23, Proposition 3.3], there exist an operator R with smooth kernel $K_R \in C^\infty(\mathbf{X}_\varepsilon \times \mathbf{X}_\varepsilon)$, and an operator A_1 in $L_{1,0}^{2m}(\mathbf{X}_\varepsilon)$ which is properly supported in \mathbf{X}_ε such that, on $L^2(\mathbf{X}_\varepsilon)$,

$$A_0 = A_1 + R.$$

A_1 is a classical pseudodifferential operator in \mathbf{X}_ε , with the same principal symbol as A_0 , and is elliptic on \mathbf{X}_ε in view of (5). Applying now the Gårding inequality as stated in [11, p. 51], one deduces the existence of a constant $C_1 > 0$ such that, for all $u \in C_c^\infty(\mathbf{X}_\varepsilon)$ with support in $\bar{\mathbf{X}}$,

$$\operatorname{Re}((A_1 + C_1 \mathbf{1})u, u)_{L^2(\mathbf{X}_\varepsilon)} \geq \frac{1}{C_1} \|u\|_{H^m(\mathbf{X}_\varepsilon)}^2.$$

Now, by the Schwarz inequality,

$$\|Ru\|_{L^2(\mathbf{X})}^2 = \int_{\mathbf{X}} |Ru(x)|^2 dx \leq \int_{\mathbf{X}} \left(\int_{\mathbf{X}} |K_R(x, y)|^2 dy \int_{\mathbf{X}} |u(z)|^2 dz \right) dx, \quad u \in C_c^\infty(\mathbf{X}),$$

which implies that the restriction of R to $L^2(\mathbf{X})$ is a bounded operator. Consequently, there exists a constant $C_2 > 0$ such that for $u \in C_c^\infty(\mathbf{X})$

$$((A_0 + C_1 \mathbf{1})u, u)_{L^2(\mathbf{X})} \geq \frac{1}{C_1} \|u\|_{H^m(\mathbf{X}_\varepsilon)}^2 + \operatorname{Re}(Ru, u)_{L^2(\mathbf{X})} \geq \frac{1}{C_1} \|u\|_{H^m(\mathbf{X})}^2 - C_2 \|u\|_{L^2(\mathbf{X})}^2,$$

and the assertion of the lemma follows. \square

Next note that if A_0 were properly supported, then $A_0 \circ \operatorname{ext}: C_c^\infty(\mathbf{X}) \rightarrow C_c^\infty(\mathbf{X}_1)$, where \mathbf{X}_1 is some compact set in \mathbb{R}^n , see [23, Proposition 3.4]. By continuity, this map would extend to a map from $\mathcal{D}'(\mathbf{X})$ to $\mathcal{D}'(\mathbf{X}_1)$, but in general it is not immediately clear if the restriction of A_0 to \mathbf{X} extends to $\mathcal{D}'(\mathbf{X})$. Nevertheless, as a pseudodifferential operator in the class $\Pi_{1,0}^{2m}(\mathbb{R}^n)$, the operator $A_0: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ extends to a mapping from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, see [14]. Therefore, if $u \in L^2(\mathbf{X})$, then $(A_0 \circ \operatorname{ext})(u) \in \mathcal{S}'(\mathbb{R}^n)$, and via the inclusion $\mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbf{X})$, the operator $\operatorname{res} \circ A_0 \circ \operatorname{ext}$ extends naturally to an operator from $L^2(\mathbf{X})$ to $\mathcal{D}'(\mathbf{X})$. Let us now assume that A_0 is symmetric, and that (3) is satisfied. Under these circumstances, the previous lemma implies that the operator

$$\operatorname{res} \circ A_0 \circ \operatorname{ext}: C_c^\infty(\mathbf{X}) \rightarrow L^2(\mathbf{X})$$

is lower semi-bounded, and we shall denote its Friedrichs extension by A . It is a self-adjoint operator in $L^2(\mathbf{X})$, and is itself lower semi-bounded. Its spectrum is real. The following proposition shows that A has compact resolvent, which implies that the spectrum of A is discrete, i.e. it consists of a sequence of isolated eigenvalues of finite multiplicity tending to infinity, while the essential spectrum of A is empty.

Proposition 1. *As an operator in $L^2(\mathbf{X})$, A has compact resolvent. Moreover, $D(A) \subset H_0^m(\mathbf{X})$ and*

$$((A + M)u, u)_{L^2(\mathbf{X})} \geq C \|u\|_{H^m(\mathbf{X})}^2 \quad \forall u \in D(A). \quad (6)$$

Here, $H_0^m(\mathbf{X})$ denotes the closure of $C_c^\infty(\mathbf{X})$ in $H^m(\mathbf{X}) = \{u \in \mathcal{D}'(\mathbf{X}): \partial^\alpha u \in L^2(\mathbf{X}), |\alpha| \leq m\}$ with respect to the Sobolev norm.

Proof. Put $\tilde{A} = \text{res} \circ A_0 \circ \text{ext} : C_c^\infty(\mathbf{X}) \rightarrow L^2(\mathbf{X})$. In view of (4), \tilde{A} is semi-bounded on $C_c^\infty(\mathbf{X})$. Let $\mathcal{Q}(A)$ be its form domain, that is, the completion of $C_c^\infty(\mathbf{X})$ with respect to the norm $p(v) = \sqrt{((\tilde{A} + M)v, v)}$, see [22, p. 177]. $\mathcal{Q}(A)$ is endowed with the limit norm $\|\cdot\|_{\mathcal{Q}(A)}$ of p . According to (4), $\mathcal{Q}(A) \subset H_0^m(\mathbf{X})$. Since A is the Friedrichs extension of \tilde{A} , one has $D(A) \subset \mathcal{Q}(A)$, and we obtain Eq. (6). Let now $\lambda < -M$. If $u \in D(A)$, the Schwarz inequality yields

$$\|u\|_{H^m(\mathbf{X})} \leq C \|(A - \lambda)u\|_{L^2(\mathbf{X})}$$

for some constant $C > 0$. Thus, if $v \in L^2(\mathbf{X})$, $\|(A - \lambda)^{-1}v\|_{H^m(\mathbf{X})} \leq C\|v\|_{L^2(\mathbf{X})}$. Therefore $(A - \lambda)^{-1}$ is a continuous map from $L^2(\mathbf{X})$ to $H_0^m(\mathbf{X})$. But the injection $H_0^m(\mathbf{X}) \hookrightarrow L^2(\mathbf{X})$ is compact by the Rellich theorem. Consequently, A must have compact resolvent. \square

Consider now a compact group of isometries $G \subset O(n)$ acting on Euclidean space \mathbb{R}^n , and assume that the bounded domain \mathbf{X} in \mathbb{R}^n is invariant under G . Then its boundary is G -invariant, too. Let T be the unitary representation of G in the Hilbert spaces $L^2(\mathbb{R}^n)$ and $L^2(\mathbf{X})$ defined in (1), and assume that the operator A_0 commutes with the representation T . The G -action on \mathbf{X} induces a Hamiltonian action of G in the cotangent bundle $T^*(\mathbf{X})$ of \mathbf{X} given by

$$G \times T^*(\mathbf{X}) \rightarrow T^*(\mathbf{X}) : (k, x, \xi) \rightarrow \sigma_k(x, \xi) = (\kappa_k(x), {}^t\kappa'_k(x)^{-1}(\xi)) = (\kappa_k(x), \kappa_k(\xi)),$$

where we wrote $\kappa_k(x) = kx$. Now, since

$$T(k)\text{Op}^\tau(a)T(k^{-1}) = \text{Op}^\tau(a \circ \sigma_k), \quad a \in S(\tilde{g}, p),$$

the G -invariance of A_0 is equivalent to the G -invariance of its symbol, by the uniqueness of the τ -symbol. In particular, the principal symbol a_{2m} of A_0 is invariant under σ_k for all $k \in G$. Since the operator A is also G -invariant, the eigenspaces of A are unitary G -modules that decompose into irreducible subspaces. The restriction of A to the isotypic component $\text{res } \mathcal{H}_\chi$ in the Peter-Weyl decomposition of $(T, L^2(\mathbf{X}))$ is called the reduced operator, and is denoted by A_χ . Its domain is $D(A_\chi) = D(A) \cap \text{res } \mathcal{H}_\chi$. As explained in [4], A_χ inherits from A the property of having compact resolvent, and the spectrum of A is equal to the union over χ in \hat{G} of the spectra of the operators A_χ .

Our purpose in this paper is to investigate the spectral counting function $N_\chi(\lambda)$ of A_χ , which is given by the number of eigenvalues of A_χ , counting multiplicities, that are equal or less than $\lambda \in \mathbb{R}$. It corresponds exactly the number of eigenvalues of A equal or less than λ , whose eigenfunctions belong to the χ -isotypic component of $L^2(\mathbf{X})$, so that

$$N_\chi(\lambda) = d_\chi \sum_{t \leq \lambda} \mu_\chi(t),$$

where $\mu_\chi(t)$ denotes the multiplicity of any irreducible representation of dimension d_χ corresponding to the character χ in the eigenspace of A with eigenvalue t . We shall study $N_\chi(\lambda)$ using the method of approximate spectral projection operators, which was first introduced by Shubin and Tulovskii, and adapted to the case of bounded domains by Levendorskii. It departs from the observation that

$$N(\lambda) = \text{tr}(E_\lambda),$$

where $E_\lambda = \mathbf{1}_{(-\infty, \lambda]}(A)$ is the spectral projector of A belonging to the value λ . The idea is then to approximate E_λ by a pseudodifferential operator \mathcal{E}_λ . The trace of \mathcal{E}_λ should then give a good approximation of $N(\lambda)$. The approximate spectral projection operators \mathcal{E}_λ will be constructed using Weyl quantization. In order to define them, we introduce now the relevant symbols. Thus, let $a_\lambda \in S(g, 1)$, and $d \in S(g, d)$ be G -invariant symbols which, on $\mathbf{X}_\varrho \times \{\xi: |\xi| > 1\}$, $\mathbf{X}_\varrho = \{x: \text{dist}(x, \mathbf{X}) < \varrho\}$, are given by

$$a_\lambda(x, \xi) = \frac{1}{1 + \lambda|\xi|^{-2m}} \left(1 - \frac{\lambda}{a_{2m}(x, \xi)} \right),$$

$$d(x, \xi) = |\xi|^{-1},$$

where $\varrho > 0$ is some fixed constant, and in addition assume that d is positive and that $d(x, \xi) \rightarrow 0$ as $|x| \rightarrow \infty$. We also define

$$b_\lambda(x, \xi) = a_\lambda(x, \lambda^{1/2m} \xi),$$

which for $|\xi| > \lambda^{-1/2m}$ is given by

$$b_\lambda(x, \xi) = \frac{1}{1 + |\xi|^{-2m}} \left(1 - \frac{1}{a_{2m}(x, \xi)} \right), \quad (7)$$

and actually independent of λ . We need to define smooth approximations to the Heaviside function, and to certain characteristic functions on \mathbf{X} . Thus, let $\tilde{\chi}$ be a smooth function on the real line satisfying $0 \leq \tilde{\chi} \leq 1$, and

$$\tilde{\chi}(s) = \begin{cases} 1 & \text{for } s < 0, \\ 0 & \text{for } s > 1. \end{cases}$$

Let $C_0 > 0$ and $\delta \in (1/4, 1/2)$ be constants, and put $\omega = 1/2 - \delta$. We then define the G -invariant functions

$$\chi_\lambda = \tilde{\chi} \circ ((a_\lambda + 4h^{\delta-\omega} + 8C_0d)h^{-\delta}), \quad \chi_\lambda^+ = \tilde{\chi} \circ ((a_\lambda - 4h^{\delta-\omega} - 8C_0d)h^{-\delta}),$$

where $0 < \delta - \omega < 1/2$. One can then show that $\chi_\lambda, \chi_\lambda^+ \in S(h^{-2\delta}g, 1) = \Gamma_{1-\delta, \delta}^0(\mathbb{R}^{2n})$ uniformly in λ , see Part I, Lemma 10. Next, let U be a subset in \mathbb{R}^{2n} , $c > 0$, and put

$$U(c, g) = \{(x, \xi) \in \mathbb{R}^{2n}: \exists (y, \eta) \in U, \ g_{(x, \xi)}(x - y, \xi - \eta) < c\};$$

according to [18, Corollary 1.2], there exists a smoothened characteristic function $\psi_c \in S(g, 1)$ belonging to the set U and the parameter c , such that $\text{supp } \psi_c \subset U(2c, g)$, and $\psi_c|_{U(c, g)} = 1$. Let now

$$\mathcal{M}_\lambda = \{(x, \xi) \in \mathbb{R}^{2n}: a_\lambda < 4h^{\delta-\omega} + 8C_0d\}.$$

Both \mathcal{M}_λ and $\partial\mathbf{X} \times \mathbb{R}^n$ are invariant under σ_k for all $k \in G$, as well as $(\partial\mathbf{X} \times \mathbb{R}^n)(c, h^{-2\delta}g)$, and $\mathcal{M}_\lambda(c, h^{-2\delta}g)$, due to the invariance of $a_{2m}(x, \xi)$, and the considered metrics and symbols.

Now, let $\tilde{\eta}_c, \psi_{\lambda,c} \in S(h^{-2\delta}g, 1)$ be smoothened characteristic functions corresponding to the parameter c , and the sets $\partial\mathbf{X} \times \mathbb{R}^n$ and \mathcal{M}_λ , respectively. According to Lemma 5 in Part I, we can assume that they are invariant under σ_k for all $k \in G$; otherwise consider $\int_G \tilde{\eta}_c \circ \sigma_k dk$, $\int_G \psi_{\lambda,c} \circ \sigma_k dk$, respectively. We then define the functions

$$\eta_{\lambda,-c}(x, \xi) = \begin{cases} 0, & x \notin \mathbf{X}, \\ (1 - \tilde{\eta}_c(x, \xi))\psi_{\lambda,1/c}(x, \xi), & x \in \mathbf{X}, \end{cases}$$

$$\eta_c(x, \xi) = \begin{cases} \tilde{\eta}_c(x, \xi), & x \notin \mathbf{X}, \\ 1, & x \in \mathbf{X}. \end{cases}$$

Only the support of $\psi_{\lambda,c}$ depends on λ , but not its growth properties, so that $\eta_c, \eta_{\lambda,-c} \in S(h^{-2\delta}g, 1)$ uniformly in λ . Furthermore, since $\tilde{\eta}_{2c} = 1$ on $\text{supp } \tilde{\eta}_c$, and $\psi_{\lambda,1/c} = 1$ on $\text{supp } \psi_{\lambda,1/2c}$, one has $\eta_{\lambda,-c} = 1$ on $\text{supp } \eta_{\lambda,-2c}$, which implies $\eta_{\lambda,-2c}\eta_{\lambda,-c} = \eta_{\lambda,-2c}$. Similarly, one verifies $\eta_c\eta_{2c} = \eta_c$. We are now ready to define the approximate spectral projection operators.

Definition 2. The approximate spectral projection operators of the first and second kind are defined by the equations

$$\tilde{\mathcal{E}}_\lambda = \text{Op}^w(\eta_{\lambda,-2})\text{Op}^w(\chi_\lambda)\text{Op}^w(\eta_{\lambda,-2}), \quad \mathcal{E}_\lambda = \tilde{\mathcal{E}}_\lambda^2(3 - 2\tilde{\mathcal{E}}_\lambda),$$

while the ones of the third and fourth kind are given by

$$\tilde{\mathcal{F}}_\lambda = \text{Op}^w(\eta_2^2\chi_\lambda^+), \quad \mathcal{F}_\lambda = \tilde{\mathcal{F}}_\lambda^2(3 - 2\tilde{\mathcal{F}}_\lambda).$$

Both \mathcal{E}_λ and \mathcal{F}_λ are integral operators with kernels in $\mathcal{S}(\mathbb{R}^{2n})$. By [14, Lemma 7.2], this implies that \mathcal{E}_λ and \mathcal{F}_λ are of trace class and, in particular, compact operators in $L^2(\mathbb{R}^n)$. In addition, by Theorem 2, and the asymptotic expansion (10) in Part I, one has $\sigma^\tau(\mathcal{E}_\lambda), \sigma^\tau(\mathcal{F}_\lambda) \in S(h^{-2\delta}g, 1)$ uniformly in λ . On the other hand, all the involved symbols are real valued, which by general Weyl calculus implies that $\text{Op}^w(\eta_{\lambda,-2}), \text{Op}^w(\chi_\lambda), \text{Op}^w(\eta_2^2\chi_\lambda^+)$, and consequently also \mathcal{E}_λ , and \mathcal{F}_λ , are self-adjoint operators in $L^2(\mathbb{R}^n)$. Let P_χ denote the orthogonal projector defined in (2) onto the isotypic component of the Peter–Weyl decomposition of $(T, L^2(\mathbb{R}^n))$ corresponding to the character χ . By construction, both \mathcal{E}_λ and \mathcal{F}_λ commute with the projection P_χ , so that $P_\chi\mathcal{E}_\lambda$ and $P_\chi\mathcal{F}_\lambda$ are self-adjoint operator of trace class as well. Although the decay properties of $\sigma^\tau(\mathcal{E}_\lambda), \sigma^\tau(\mathcal{F}_\lambda)$ are independent of λ , their supports do depend on λ , which will result in estimates for the trace of $P_\chi\mathcal{E}_\lambda$ and $P_\chi\mathcal{F}_\lambda$ in terms of λ that will be used in order to prove Theorem 8. In particular, the estimate for the remainder term in Theorem 8 is determined by the particular choice of the range $(1/4, 1/2)$ for the parameter δ , which guarantees that $1 - \delta > \delta$.

The method of approximate spectral projection operators is based on variational arguments. Thus, if S is a symmetric, lower semi-bounded operator in a separable Hilbert space, and if V is a subspace contained in its domain $\mathcal{D}(S)$, the variational quantity

$$\mathcal{N}(S, V) = \sup_{L \subset V} \{ \dim L : (Su, u) < 0 \quad \forall 0 \neq u \in L \}$$

can be used to give a qualitative description of the spectrum of S . In our case one has

$$N_\chi(\lambda) = \mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap C_c^\infty(\mathbf{X})).$$

Indeed, the Friedrichs extension of $\text{res} \circ A_0 \circ \text{ext} : C_c^\infty(\mathbf{X}) \cap \mathcal{H}_\chi \rightarrow \text{res } \mathcal{H}_\chi$ is given by A_χ , and the assertion follows with [18, Lemma A.2]. Now by the general theory of compact, self-adjoint operators, zero is the only accumulation point of the point spectra of \mathcal{E}_λ and \mathcal{F}_λ , as well as the only point that could possibly belong to the continuous spectrum. Therefore the number of eigenvalues of \mathcal{E}_λ which are $\geq 1/2$, and whose eigenfunctions belong to the isotypic component \mathcal{H}_χ is clearly finite, and shall be denoted by $N_\chi^{\mathcal{E}_\lambda}$. Similarly, the number of eigenvalues of the operators \mathcal{F}_λ which are $\geq 1/2$, and whose eigenfunctions belong to the isotypic component \mathcal{H}_χ , shall be denoted by $N_\chi^{\mathcal{F}_\lambda}$. As it was shown in Part I, Theorems 4 and 5, these quantities constitute upper and lower bounds for the spectral counting function $N_\chi(\lambda)$, namely

$$N_\chi^{\mathcal{E}_\lambda} - C \leq \mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap C_c^\infty(\mathbf{X})) \leq N_\chi^{\mathcal{F}_\lambda} + C$$

for some constant $C > 0$. Furthermore, by Lemmata 11 and 12 of Part I one has

$$\begin{aligned} 2 \operatorname{tr}(P_\chi \mathcal{E}_\lambda \cdot P_\chi \mathcal{E}_\lambda) - \operatorname{tr} P_\chi \mathcal{E}_\lambda - c_1 &\leq N_\chi^{\mathcal{E}_\lambda} \leq 3 \operatorname{tr} P_\chi \mathcal{E}_\lambda - 2 \operatorname{tr}(P_\chi \mathcal{E}_\lambda \cdot P_\chi \mathcal{E}_\lambda) + c_2, \\ 2 \operatorname{tr}(P_\chi \mathcal{F}_\lambda \cdot P_\chi \mathcal{F}_\lambda) - \operatorname{tr} P_\chi \mathcal{F}_\lambda - c_1 &\leq N_\chi^{\mathcal{F}_\lambda} \leq 3 \operatorname{tr} P_\chi \mathcal{F}_\lambda - 2 \operatorname{tr}(P_\chi \mathcal{F}_\lambda \cdot P_\chi \mathcal{F}_\lambda) + c_2, \end{aligned}$$

for some constants $c_i > 0$. The study of the asymptotic behaviour of $N_\chi(\lambda)$ is therefore reduced to an examination of the traces of $P_\chi \mathcal{E}_\lambda$ and $P_\chi \mathcal{F}_\lambda$, together with their squares, and will occupy us for the rest of this paper.

3. Compact group actions and the principle of the stationary phase

In this section, we shall begin to estimate the traces of $P_\chi \mathcal{E}_\lambda$ and $P_\chi \mathcal{F}_\lambda$ using the method of the stationary phase, in order to obtain a description of the spectral counting function $N_\chi(\lambda)$ as $\lambda \rightarrow +\infty$. As mentioned in the introduction, first order asymptotics for invariant elliptic operators were already obtained in [3,6] in the general case of effective group actions by using heat kernel methods; nevertheless, estimates for the remainder are not accessible via this approach. On the other hand, the derivation of remainder estimates within the framework of Fourier integral operators or, as we shall see, within the setting of approximate spectral projections, meets with serious difficulties when singular orbits are present. The reason for this is that, using these approaches, one is led to the study of the asymptotic behavior of integrals of the form

$$\int_G \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-kx)\xi/\mu} a(x, \xi, k) dx d\xi dk, \quad \mu \rightarrow 0^+, \quad (8)$$

via the generalized stationary phase theorem, where $a(x, \xi, k) \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n \times G)$ is an amplitude which might also depend on μ . While for free group actions, the critical set of the phase function $(x - kx)\xi$ is a smooth manifold, this is no longer the case for general effective actions, so that, a priori, the principle of the stationary phase cannot be applied in this case. Nevertheless, in what follows, we shall show how to circumvent this obstacle by partially resolving the singularities of the critical set of the phase function in question, and in this way obtain remainder

estimates for $N_\chi(\lambda)$ in the case of singular group actions. Let us begin by stating the generalized stationary phase theorem.

Theorem 1 (Generalized stationary phase theorem for manifolds). *Let M be a n -dimensional Riemannian manifold, $\psi \in C^\infty(M)$ be a real-valued phase function, $a \in C_c^\infty(M)$, $\mu > 0$, and set*

$$I(\mu) = \int_M e^{i\psi(m)/\mu} a(m) dm,$$

where dm denotes the volume form on M . Let $\mathcal{C} = \{m \in M: \psi': TM_m \rightarrow T\mathbb{R}_{\psi(m)} \text{ is zero}\}$ be the critical set of the phase function ψ , and assume that:

- (i) \mathcal{C} is a smooth submanifold of M of dimension p in a neighborhood of the support of a ;
- (ii) for all $m \in \mathcal{C}$, the restriction $\psi''(m)|_{N_m\mathcal{C}}$ of the Hessian of ψ at the point m to the normal space $N_m\mathcal{C}$ is a non-degenerate quadratic form.

Then, for all $N \in \mathbb{N}$, there exists a constant $C_{N,\psi} > 0$ such that

$$\left| I(\mu) - e^{i\psi_0/\mu} (2\pi\mu)^{\frac{n-p}{2}} \sum_{j=0}^{N-1} \mu^j L_j(\psi; a) \right| \leq C_{N,\psi} \mu^N \text{vol}(\text{supp } a \cap \mathcal{C}) \sup_{l \leq 2N} \|D^l a\|_{\infty, M},$$

where D^l is a differential operator on M of order l , and ψ_0 is the constant value of ψ on \mathcal{C} . Furthermore, for each j there exists a constant $\tilde{C}_{j,\psi} > 0$ such that

$$|L_j(\psi; a)| \leq \tilde{C}_{j,\psi} \text{vol}(\text{supp } a \cap \mathcal{C}) \sup_{l \leq 2j} \|D^l a\|_{\infty, \mathcal{C}},$$

and, in particular,

$$L_0(\psi; a) = \int_{\mathcal{C}} \frac{a(x)}{|\det \psi''(m)|_{N_m\mathcal{C}}|^{1/2}} d\sigma_{\mathcal{C}}(m) e^{i\pi\sigma_{\psi''}},$$

where $\sigma_{\psi''}$ is the constant value of the signature of $\psi''(m)|_{N_m\mathcal{C}}$ for m in \mathcal{C} .

Proof. See [15, Theorem 7.7.5], and [5, Theorem 3.3]. \square

From now on, we shall restrict ourselves to the study of $\text{tr } P_\chi \mathcal{E}_\lambda$, since the corresponding considerations for \mathcal{F}_λ are completely analogous. Let therefore $\sigma^l(\mathcal{E}_\lambda)(x, \xi)$ denote the left symbol of \mathcal{E}_λ . Since $\sigma^l(\mathcal{E}_\lambda)$ is G -invariant, we have

$$P_\chi \mathcal{E}_\lambda u(x) = d_\chi \int_G \iint \overline{\chi(k)} e^{i(x-ky)\xi} \sigma^l(\mathcal{E}_\lambda)(x, \xi) u(y) dy d\xi dk, \quad u \in C_c^\infty(\mathbb{R}^n).$$

The kernel of $P_\chi \mathcal{E}_\lambda$, which is a rapidly decreasing function, is given by the absolutely convergent integral

$$K_{P_\chi \mathcal{E}_\lambda}(x, y) = d_\chi \int_G \int \overline{\chi(k)} e^{i(x-ky)\xi} \sigma^l(\mathcal{E}_\lambda)(x, \xi) d\xi dk.$$

Consequently, the trace of $P_\chi \mathcal{E}_\lambda$ can be computed by

$$\mathrm{tr} P_\chi \mathcal{E}_\lambda = \int K_{P_\chi \mathcal{E}_\lambda}(x, x) dx = d_\chi \int_G \int \int \overline{\chi(k)} e^{i(x-kx)\xi} \sigma^l(\mathcal{E}_\lambda)(x, \xi) dx d\xi dk.$$

As already noticed, the decay properties of $\sigma^l(\mathcal{E}_\lambda) \in S(h^{-2\delta}g, 1) = \Gamma_{1-\delta, \delta}^0(\mathbb{R}^{2n})$ are independent of λ , while its support does depend on λ . Indeed, as it was already explained in Part I, Eq. (51),

$$\sigma^l(\mathcal{E}_\lambda) = (\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda) + f_\lambda + r_\lambda, \quad (9)$$

where $r_\lambda \in S(h^{-2\delta}g, h^{N(1-2\delta)})$ for arbitrary large N , and $f_\lambda \in S(h^{-2\delta}g, h^{1-2\delta})$, everything uniformly in λ . Moreover, in Lemma 9 we shall see that

$$\mathrm{supp} f_\lambda \subset A_{c, \lambda} = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : a_\lambda < c(h^{\delta-\omega} + d)\}.$$

Now, since $|r_\lambda(x, \xi)| \leq C'(1 + |x|^2 + |\xi|^2)^{-N/2}$ for some constant C' independent of λ and N arbitrarily large, we get the uniform bound

$$\iint |r_\lambda(x, \xi)| dx d\xi \leq C;$$

note that the x -dependence of $h(x, \xi)$ is crucial at this point. In order to determine the asymptotic behaviour of $\mathrm{tr} P_\chi \mathcal{E}_\lambda$ with respect to λ , we can therefore neglect the contribution coming from $r_\lambda(x, \xi)$, so that

$$\mathrm{tr} P_\chi \mathcal{E}_\lambda = d_\chi \int_G \int \int \overline{\chi(k)} e^{i(x-kx)\xi} [(\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda) + f_\lambda](x, \xi) dx d\xi dk + O(1),$$

as λ goes to infinity. To apply the generalized stationary phase theorem, we introduce the new parameter

$$\mu = \lambda^{-1/2m}, \quad \lambda = \mu^{-2m},$$

and performing the change of variables

$$\Psi_\mu : (x, \xi) \mapsto (x, \mu\xi)$$

we obtain

$$\mathrm{tr} P_\chi \mathcal{E}_\lambda = d_\chi \lambda^{n/2m} I(\lambda^{-1/2m}) + O(1),$$

where we set

$$\begin{aligned}
 I(\mu) &= \int_G \int_{\mathbf{X}} \int_{\mathbb{R}^n} e^{\frac{i}{\mu} \psi(x, \xi, k)} \overline{\chi(k)} \sigma_\mu(x, \xi) dx d\xi dk, \\
 \psi(x, \xi, k) &= (x - kx)\xi, \\
 \sigma_\mu &= [(\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda) + f_\lambda] \circ \Psi_\mu^{-1}.
 \end{aligned} \tag{10}$$

As we shall see later, there exists a compact subset $K \subset \mathbb{R}^{2n}$, such that σ_μ has support in K for all $\mu > 0$, see (38). To get an asymptotic expansion of $I(\mu)$ as $\mu \rightarrow 0^+$ via the generalized stationary phase theorem, we commence by examining the critical set

$$\mathcal{C} = \{(x, \xi, k) \in \mathbf{X} \times \mathbb{R}^n \times G : \psi'(x, \xi, k) = 0\} \tag{11}$$

of the phase function ψ . After a straightforward computation we obtain

$$\mathcal{C} = \{(z, k) \in \Omega_0 \times G : kz = z\},$$

where we put $z = (x, \xi)$, and

$$\Omega_0 = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : \langle xA, \xi \rangle = 0 \text{ for all } A \in \mathfrak{g}\},$$

\mathfrak{g} being the Lie algebra of G . $\langle \cdot, \cdot \rangle$ denotes the Euclidean product in \mathbb{R}^n . Note that Ω_0 is invariant under the Hamiltonian action of G on the cotangent space $T^*(\mathbf{X})$ given by $(x, \xi) \mapsto (kx, k\xi)$, as well as homogeneous with respect to x and ξ . It has the following interpretation in terms of the Hamiltonian action of G on $T^*(\mathbf{X})$. If (A_1, \dots, A_d) is a basis of \mathfrak{g} , let

$$\mathbb{J} : T^*(\mathbf{X}) \simeq \mathbf{X} \times \mathbb{R}^n \rightarrow \mathfrak{g} \simeq \mathbb{R}^d, \quad (x, \xi) \rightarrow (\langle A_1 x, \xi \rangle, \dots, \langle A_d x, \xi \rangle),$$

be the associated momentum map, and denote by

$$\Omega_0/G = \mathbb{J}^{-1}(\{0\})/G$$

the symplectic quotient of $T^*(\mathbf{X})$ at level zero. This quotient is naturally related to the critical set of the phase function in question, and we shall prove that $N_\chi(\lambda)$ is asymptotically determined by a certain volume of the quotient Ω_0/G . Now, the major difficulty in applying the generalized stationary phase theorem in our setting stems from the fact that, due to the singular orbit structure of the underlying group action, the zero level Ω_0 of the momentum map, and, consequently, the considered critical set \mathcal{C} , are in general singular varieties. In fact, if the G -action on $T^*(\mathbf{X})$ is not free, the considered momentum map is no longer a submersion, so that Ω_0 and Ω_0/G are not smooth anymore. To circumvent this difficulty, we will partially resolve the singularities of \mathcal{C} by constructing a partial resolution of Ω_0 , which takes into account the singular orbit structure of the underlying G -action, and then apply the stationary phase theorem in the resolution space.²

² As we shall see in Section 5,

$$\sigma_\mu(x, \xi) \rightarrow \mathbf{1}_{\{a_{2m} \leq 1\}}(x, \xi) \quad \text{as } \mu \rightarrow 0^+,$$

In what follows, we shall therefore briefly recall some basic notions of the theory of compact group actions. For a detailed exposition, we refer the reader to [1] or [17]. Let G be a compact Lie group acting locally smoothly on some n -dimensional C^∞ -manifold M , and denote the stabilizer, or isotropy group, of $x \in M$ by

$$G_x = \{k \in G: k \cdot x = x\}.$$

The orbit of a point $x \in M$ under the action of G will be denoted by \mathcal{O}_x . Assume that M/G is connected. One of the main results in the theory of compact group actions is the following.

Theorem 2 (Principal orbit theorem). *There exists a maximum orbit type G/H for G on M . The union $M_{(H)}$ of orbits of type G/H is open and dense, and its image in M/G is connected.*

Proof. See [1, Theorem IV.3.1]. \square

Orbits of type G/H are called of principal type, and the corresponding isotropy groups are called principal. A principal isotropy group has the property that it is conjugated to a subgroup of each stabilizer of M . The following result says that there is a stratification of G -spaces into orbit types.

Theorem 3. *Let G and M be as above, K a subgroup of G , and denote the set of points on orbits of type G/K by $M_{(K)}$. Then $M_{(K)}$ is a topological manifold, which is locally closed. Furthermore, $M_{(K)}$ consists of orbits of type less than or equal to type G/K . The orbit map $M_{(K)} \rightarrow M_{(K)}/G$ is a fiber bundle projection with fiber G/K and structure group $N(K)/K$.*

Proof. See [1, Theorem IV.3.3]. \square

Let now M_τ denote the union of non-principal orbits of dimension at most τ .

Theorem 4. *If κ is the dimension of a principal orbit, then $\dim M/G = n - \kappa$, and M_τ is a closed set of dimension at most $n - \kappa + \tau - 1$.*

Proof. See [1, Theorem IV.3.8]. \square

Here the dimension of M_τ is understood in the sense of general dimension theory. In what follows, we shall write $\text{Sing } M = M - M_{(H)} = M_\kappa$. Clearly,

$$\text{Sing } M = M_0 \cup (M_1 - M_0) \cap (M_2 - M_1) \cup \dots \cup (M_\kappa - M_{\kappa-1}),$$

where $M_i - M_{i-1}$ is precisely the union of non-principal orbits of dimension i , and $M_{-1} = \emptyset$, by definition. Note that

where $\mathbf{1}_A$ stands for the characteristic function of the set A . By homogeneity, $a_{2m}(0, 0) = 0$, so that zero is contained in the support of $\mathbf{1}_{\{a_{2m} \leq 1\}}$. In general, σ_μ is therefore not supported away from the set of singular points of \mathcal{C} , since $(0, 0)$ is always a singularity of Ω_0 in case that $0 \in \mathbf{X}$.

$$M_i - M_{i-1} = \bigcup_j M_{(H_j^i)}, \quad H_j^i \subset G, \quad \dim G/H_j^i = i,$$

is a disjoint union of topological manifolds of possibly different dimensions. We apply this theory now to the case where $M = \mathbb{R}^n$, and G is a compact subgroup of $O(n)$.

Definition 3. Let G/H_0 be the principal orbit type of the action of $G \subset O(n)$ on \mathbb{R}^n , and denote by κ the dimension of G/H_0 .

Since \mathbf{X} is open in \mathbb{R}^n , it has the same principal orbit type than \mathbb{R}^n . Now, even if Ω_0 is not a smooth manifold, it can be shown that it has a Whitney stratification into smooth submanifolds, see [20, Theorem 8.3.1], which corresponds to the stratification of $T^*(\mathbf{X})$ and \mathbb{R}^n into orbit types. In particular, the strata of Ω_0 are submanifolds of \mathbb{R}^{2n} , and Ω_0 admits a principal orbit type, too.

Proposition 2. Let $\text{Reg } \Omega_0 = \Omega_{0(H_1)}$ be the principal stratum of Ω_0 . Then $\text{Reg } \Omega_0$ is an open dense subset of Ω_0 , and a submanifold of $\mathbf{X} \times \mathbb{R}^n$ of codimension κ . Moreover, for $z \in \text{Reg } \Omega_0$ one has

$$T_z(\text{Reg } \Omega_0) = (Jgz)^\perp, \quad \text{where } J = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}. \quad (12)$$

Futhermore, H_1 is conjugated to H_0 , and thus

$$\text{Reg } \Omega_0 = \{z \in \Omega_0: G_z \text{ is conjugated to } H_0\}.$$

In particular, if $(x, \xi) \in \Omega_0$, and if \mathcal{O}_x or \mathcal{O}_ξ are of type G/H_0 , then $(x, \xi) \in \text{Reg } \Omega_0$.

To prove the proposition, we need the following.

Lemma 2. Assume that $(x, \xi) \in \Omega_0$. If \mathcal{O}_x is of principal orbit type in \mathbb{R}^n , then $G_x \subset G_\xi$. If \mathcal{O}_ξ is of principal orbit type in \mathbb{R}^n , then $G_\xi \subset G_x$.

Proof. Let $(x, \xi) \in \Omega_0$, that is, $\xi \in N_x \mathcal{O}_x$, where $N_x \mathcal{O}_x$ denotes the normal space to the G -orbit \mathcal{O}_x at the point x , which is a vector subspace in \mathbb{R}^n . Assume now that \mathcal{O}_x is of principal type. Denote by V_ε the open ε -ball in $N_x \mathcal{O}_x$, and consider the linear tube

$$G \times_{G_x} V_\varepsilon \rightarrow G \cdot V_\varepsilon, \quad [g, v] \rightarrow gv,$$

around \mathcal{O}_x , see [1, Corollary II.5.2]. By [1, Theorem IV.3.2], G_x acts trivially on V_ε , and consequently also on $N_x \mathcal{O}_x$, and the assertion follows. To see this directly, one can also argue as follows. Let $(x, \xi) \in \Omega_0$, so that $\xi \in (\mathfrak{g}x)^\perp$. If $g \in G_x$, then $g\xi \in (\mathfrak{g}x)^\perp$. Thus $(g - \mathbf{1})\xi \in (\mathfrak{g}x)^\perp$. We claim that if \mathcal{O}_x is of principal orbit type in \mathbb{R}^n , then $(g - \mathbf{1})\xi \in \mathfrak{g}x$, which will yield $(g - \mathbf{1})\xi = 0$, and prove the inclusion $G_x \subset G_\xi$. Now, by [17, Theorem 4.19], the canonical projection $\pi: \mathbb{R}_{(H_0)}^n \rightarrow \mathbb{R}_{(H_0)}^n/G$ is a smooth submersion. Since the preimage of the tangent space of a smooth manifold under a submersion is equal to the tangent space of the preimage of the considered manifold at the given point, $\ker d_x \pi = \mathfrak{g}x$. Moreover, since $M_{(H_0)}$ is an open set of \mathbb{R}^n , one can differentiate the relation

$$\pi(gy) = \pi(y),$$

with respect to variable y at x to obtain $d_{gx}\pi \circ g = d_x\pi$. Since $gx = x$, $d_x\pi \circ (g - 1) = 0$. This proves that the image of $g - 1$ is contained in $\ker d_x\pi = \mathfrak{g}x$. \square

Proof of Proposition 2. The first part of the statement follows from the references previously given, while the characterization of the tangent space is obtained by observing that $\dim \text{Reg } \Omega_0 = 2n - \kappa$. By the previous lemma, $(\mathbb{R}^n \times \mathbb{R}_{(H_0)}^n) \cap \Omega_0$ is a non-empty open subset of Ω_0 consisting of orbits of type G/H_0 . As $\text{Reg } \Omega_0$ is open and dense in Ω_0 , it must intersect $(\mathbb{R}^n \times \mathbb{R}_{(H_0)}^n) \cap \Omega_0$, and therefore consist of orbits of type G/H_0 . \square

In what follows, we will denote by $\text{Sing } \Omega_0$ the complement of $\text{Reg } \Omega_0$ in Ω_0 . The next lemma will provide us with a suitable parametrization of $\text{Reg } \Omega_0$.

Lemma 3. *The sets $\{(x, \xi) \in \text{Reg } \Omega_0: x \in \text{Sing } \mathbb{R}^n\}$, $\{(x, \xi) \in \text{Reg } \Omega_0: \xi \in \text{Sing } \mathbb{R}^n\}$ have measure zero in $\text{Reg } \Omega_0$ with respect to the induced volume form on $\text{Reg } \Omega_0$.*

Proof. We shall show that $N = \{(x, \xi) \in \Omega_0: x \in \text{Sing } \mathbb{R}^n\}$ is a closed set in Ω_0 of dimension at most $2n - \kappa - 1$. Indeed, with $M = \mathbb{R}^n$, and notations as above,

$$N = \bigcup_{i=0}^{\kappa} \{(x, \xi) \in \Omega_0: x \in M_i - M_{i-1}\} = \bigcup_{i=0}^{\kappa} \bigcup_{j(i)} \{(x, \xi) \in \mathbb{R}^{2n}: x \in \mathbb{R}_{(H_j^i)}^n, \xi \in N_x \mathcal{O}_x\},$$

where the union over $j(i)$ ranges over all non-principal orbit types G/H_j^i with $\dim G/H_j^i = i$. By the previous theorem, $\dim \mathbb{R}_{(H_j^i)}^n \leq \dim M_i \leq n - \kappa + i - 1$, and in addition, $\dim N_x \mathcal{O}_x = n - i$ for all $x \in \mathbb{R}_{(H_j^i)}^n$. Consequently, $\{(x, \xi) \in \mathbb{R}^{2n}: x \in \mathbb{R}_{(H_j^i)}^n, \xi \in N_x \mathcal{O}_x\}$ is a subset of Ω_0 of dimension at most $2n - \kappa - 1$. Since for orthogonal group actions there are only finitely many orbit types, the union over $j(i)$ is finite, and the assertion of the lemma follows. \square

Finally, for future reference we note the following.

Lemma 4. *The set*

$$\text{Reg } \mathcal{C} = \{(z, k) \in \text{Reg } \Omega_0 \times G: kz = z\}$$

is a smooth submanifold of dimension $2n + d - 2\kappa$, and for $(z, k) \in \text{Reg } \mathcal{C}$,

$$T_{(z,k)} \text{Reg } \mathcal{C} = \{(\alpha, Ak): \alpha \in T_z \text{Reg } \Omega_0, A \in \mathcal{G} \text{ and } (1 - k)\alpha + Az = 0\}.$$

Proof. See [4, Lemma 3.2]. \square

In particular note that if (z, k) belongs to $\text{Sing } \mathcal{C}$, the complement of $\text{Reg } \mathcal{C}$ in \mathcal{C} , then z must necessarily lie in $\text{Sing } \Omega_0$. After these preliminary remarks, we are now ready for the analysis of $I(\mu)$.

4. Phase analysis and partial desingularization

We shall now start with the computation of an asymptotic formula for integrals of the form (8) via the generalized stationary phase theorem by partially resolving the singularities of the critical set

$$\mathcal{C} = \{(x, \xi, k) \in \mathbf{X} \times \mathbb{R}^n \times G : \psi'(x, \xi, k) = 0\}$$

of the phase function $\psi(x, \xi, k) = \langle x - kx, \xi \rangle$. Such a resolution will be given by a proper \mathbb{R} -analytic map $\zeta : \tilde{M} \rightarrow M$ of some smooth manifold \tilde{M} onto $M = \mathbb{R}^n$, inducing a transformation $\zeta : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ such that $\tilde{\mathcal{C}}$ is a partially desingularized subvariety of \tilde{M} , and ζ induces an isomorphism of real analytic manifolds $\zeta^{-1}(\text{Reg } \mathcal{C}) \rightarrow \text{Reg } \mathcal{C}$, where $\text{Reg } \mathcal{C}$ denotes the set of nonsingular points of \mathcal{C} . By performing such a resolution we will be led to a new phase function, whose critical set is no longer a singular variety. As before, denote by $\mathbb{R}_{(H_0)}^n$ the union of all orbits of principal type G/H_0 in \mathbb{R}^n . We will construct an explicit resolution of $\tilde{\mathcal{C}}$ by constructing a resolution of Ω_0 first, under the following assumption.

Assumption 1. The set $\text{Sing } \mathbb{R}^n = \mathbb{R}^n \setminus \mathbb{R}_{(H_0)}^n$ is included in a strict vector subspace F of \mathbb{R}^n of dimension $r < n$.

Remark 1. Particular cases of Assumption 1 are as follows.

- (i) *Transitive actions on the sphere.* For any compact subgroup of $O(n)$ acting transitively on the $(n - 1)$ -dimensional sphere, $\text{Sing } \mathbb{R}^n = \{0\}$. The list of compact, connected Lie groups acting transitively and effectively on spheres has been found by Montgomery and Samelson [19]. It includes all the holonomy groups of a simply-connected Riemannian manifold with an irreducible, nonsymmetric metric appearing in Berger's list, and in particular, the group $SO(n)$ acting on \mathbb{R}^n .
- (ii) *Cylindrical actions.* For the group of rotations around an axis in \mathbb{R}^n , $\mathbb{R}_{\text{sing}}^n$ is equal to the rotation axis. More generally, any group conjugated to $G \times \{\mathbf{1}_q\}$ in $O(n)$, where G is a compact subgroup of $O(p)$ acting transitively on the $(p - 1)$ -dimensional sphere, and $p + q = n$, is included.

We begin by considering the blowing-up of $M = \mathbb{R}^{2n}$ with center $C = \{\xi_1 = \cdots = \xi_n = 0\}$ given by

$$\tilde{M} = \{(x, \xi, [\mu]) \in M \times \mathbb{RP}^{n-1} : \xi_i \mu_j = \xi_j \mu_i, i < j\},$$

together with the monoidal transformation

$$\zeta_M : \tilde{M} \rightarrow M, \quad (x, \xi, [\mu]) \mapsto (R_0 x, R_0 \xi),$$

with $R_0 \in O(n)$ such that

$$R_0(\mathbb{R}^r \times \{0\}) = F.$$

Covering \tilde{M} with the charts $\tilde{M}_j = \tilde{M} \cap (M \times U_j)$, where $U_j = \{[\mu] \in \mathbb{RP}^{n-1} : \mu_j \neq 0\}$, one obtains in \tilde{M}_j the local coordinates

$$x_i, \quad i = 1, \dots, n, \quad \eta_k = \frac{\mu_k}{\mu_j}, \quad \eta_j = \xi_j, \quad k = 1, \dots, n,$$

and we write

$$\tilde{\varphi}_j : \mathbb{R}^{2n} \rightarrow \tilde{M}_j, \quad (x, \eta) \mapsto (x, \eta_j(\eta_1, \dots, 1, \dots, \eta_n), [\eta_1 : \dots : 1 : \dots : \eta_n]).$$

Now, the total transform of Ω_0 is given by $\tilde{\Omega}_0^{\text{tot}} = \zeta_M^{-1}(\Omega_0)$, and contains the exceptional divisor $E = \zeta_M^{-1}(C)$, while the strict transform of Ω_0 in the j th chart is locally given by

$$\tilde{\Omega}_0^{\text{st}} = \{(x, \eta) \in \mathbb{R}^{2n} : \langle AR_0x, R_0(\eta_1, \dots, 1, \dots, \eta_n) \rangle = 0, A \in \mathfrak{g}\}.$$

For $j = r + 1, \dots, n$, it is a non-singular variety, since in this case the condition $(x, \eta) \in \tilde{\Omega}_0^{\text{st}}$ implies that $(R_0x, R_0(\eta_1, \dots, 1, \dots, \eta_n)) \in \text{Reg } \Omega_0$ by Assumption 1, and Proposition 2. By functoriality, the G -action on M lifts to a G -action on \tilde{M} . To construct a partial resolution for \mathcal{C} , we put $N = M \times G$, $\tilde{N} = \tilde{M} \times G$, and $\zeta_N : \tilde{N} \rightarrow N$, $(x, \xi, [\mu], k) \mapsto (x, \xi, k)$. Using the coordinates introduced above, we see that the strict transform of \mathcal{C} with respect to ζ_N is locally given by

$$\tilde{\mathcal{C}}^{\text{st}} = \{(x, \eta, k) \in \tilde{\Omega}_0^{\text{st}} \times G : (k - \mathbf{1})R_0x = 0, (k - \mathbf{1})R_0(\eta_1, \dots, 1, \dots, \eta_n) = 0\}.$$

For $j = r + 1, \dots, n$, G acts on $\tilde{\Omega}_0^{\text{st}}$ only with one orbit type, so that in this case $\tilde{\mathcal{C}}^{\text{st}}$ must be non-singular. From now on, we shall restrict ourselves to the study of the integral $I(\mu)$ defined in (10). Since each chart \tilde{M}_j completely covers \tilde{M} except for a set of measure zero, one has

$$I(\mu) = \int_G \int_{\mathbb{R}^{2n}} e^{i\tilde{\psi}_j(x, \eta, k)/\mu} \tilde{\sigma}_{\mu, j}(x, \eta) \overline{\chi}(k) |\eta_j^{n-1}| dx d\eta dk \quad (13)$$

for arbitrary j , where we put $\tilde{\psi}_j(x, \eta, k) = \psi((\zeta_M \circ \tilde{\varphi}_j)(x, \eta), k)$, $\tilde{\sigma}_{\mu, j}(x, \eta) = (\sigma_\mu \circ \zeta_M \circ \tilde{\varphi}_j)(x, \eta)$, and took into account the fact that $|\det D(\zeta_M \circ \tilde{\varphi}_j)(x, \eta)| = |\eta_j^{n-1}|$. In what follows, we shall work in the chart $j = n$, and denote $\tilde{\psi}_n$ and $\tilde{\sigma}_{\mu, n}$ simply by $\tilde{\psi}$ and $\tilde{\sigma}_\mu$, respectively. Let us now introduce the new parameter³

$$\nu = \mu/\eta_n.$$

Defining the new phase function⁴

³ The idea of introducing the new parameter ν was taken from [7, Section 6]. Nevertheless, Helffer and El-Houakmi work in spherical variables, which leads to secondary critical points that were not explicitly taken into account in their work. Our approach does not lead to secondary critical points.

⁴ The subscript “wk” stands for “weak transform.”

$$\psi_{\text{wk}} : \mathbb{R}^n \times \mathbb{R}^{n-1} \times G \rightarrow \mathbb{R}, \quad \psi_{\text{wk}}(x, \eta', k) = \langle (\mathbf{1} - k)R_0x; R_0(\eta', 1) \rangle_{\mathbb{R}^n},$$

and taking into account (38), we write

$$I(\mu) = \frac{1}{(2\pi)^n} \int_{-E_0}^{E_0} I_{\eta_n}(\mu/\eta_n) |\eta_n|^{n-1} d\eta_n,$$

where E_0 is some suitable positive number, and

$$I_{\eta_n}(\nu) = \int_G \int_{\mathbb{R}^{2n-1}} e^{\frac{i}{\nu} \psi_{\text{wk}}(x, \eta', k)} \tilde{\sigma}_{\nu\eta_n}(x, \eta', \eta_n) \bar{\chi}(k) dx d\eta' dk. \quad (14)$$

The significance of the new phase function ψ_{wk} stems from the following proposition. It will enable us to derive an asymptotic formula for $I_{\eta_n}(\nu)$ as ν goes to zero by using the stationary phase theorem in the region where η_n is not small. Note that, in particular, Theorem 1 will allow us to handle the dependence of the amplitude $\tilde{\sigma}_\mu$ in variable $\mu = \nu\eta_n$.

Proposition 3. *Let $\mathcal{C}_{\psi_{\text{wk}}} = \{\psi'_{\text{wk}} = 0\}$ denote the critical set of ψ_{wk} . Then*

$$\mathcal{C}_{\psi_{\text{wk}}} = \{(x, \eta', k) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times G : (R_0x, R_0(\eta', 1), k) \in \text{Reg } \mathcal{C}\}.$$

It is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^{n-1} \times G$ of codimension 2κ . Moreover, at each point (x, η', k) of $\mathcal{C}_{\psi_{\text{wk}}}$, the transversal Hessian of ψ_{wk} defines a non-degenerate quadratic form on the normal space $N_{(x, \eta', k)}\mathcal{C}_{\psi_{\text{wk}}}$ of $\mathcal{C}_{\psi_{\text{wk}}}$ in $\mathbb{R}^n \times \mathbb{R}^{n-1} \times G$.

Remark 2. Note that if ψ_{wk} is regarded as a function on \tilde{N} , that is, as a function of x, η , and k , the proposition implies that its critical set is given by the strict transform $\tilde{\mathcal{C}}^{\text{st}}$ of \mathcal{C} ; moreover, its transversal Hessian does not degenerate along $\tilde{\mathcal{C}}^{\text{st}}$.

Proof of Proposition 3. We shall denote by (e_1, \dots, e_n) the canonical basis of \mathbb{R}^n . With respect to the coordinates (x, η, k) one computes

$$\begin{aligned} \partial_x \psi_{\text{wk}}(x, \eta', k) &= 0 \quad \Leftrightarrow \quad (\mathbf{1} - k^{-1})R_0(\eta', 1) = 0, \\ \partial_k \psi_{\text{wk}}(x, \eta', k) &= 0 \quad \Leftrightarrow \quad \langle AR_0x, R_0(\eta', 1) \rangle = 0, \quad \forall A \in \mathfrak{g}, \\ \partial_{\eta'} \psi_{\text{wk}}(x, \eta', k) &= 0 \quad \Leftrightarrow \quad \langle (\mathbf{1} - k)R_0x, R_0e_i \rangle = 0, \quad i = 1, \dots, n-1. \end{aligned}$$

The second equation is equivalent to the fact that $(R_0x, R_0(\eta', 1)) \in \Omega_0$. By Assumption 1, $R_0(\eta', 1) \notin F$, so that using Proposition 2, we obtain that our second equation is equivalent to the fact that $(R_0x, R_0(\eta', 1)) \in \text{Reg } \Omega_0$. Using Lemma 2, the two first equations imply that $kR_0x = R_0x$, and therefore imply the third one. Consequently, we obtain

$$\begin{aligned} \mathcal{C}_{\psi_{\text{wk}}} &= \{(x, \eta', k) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times G : (k - \mathbf{1})R_0x = 0, (k - \mathbf{1})R_0(\eta', 1) = 0, \\ &\quad (R_0x, R_0(\eta', 1)) \in \text{Reg } \Omega_0\}. \end{aligned}$$

Next, we see immediately that $\mathcal{C}_{\psi_{\text{wk}}}$ is diffeomorphic to the intersection of $\text{Reg } \mathcal{C}$ and $(R_0 \times R_0)(\{\eta_n = 1\})$. Thus, in order to show that $\mathcal{C}_{\psi_{\text{wk}}}$ is a smooth manifold, we have to prove that these two sets are transversal. Let $(z, k) = (R_0 x, R_0 \eta, k) \in \text{Reg } \mathcal{C} \cap (R_0 \times R_0)(\{\eta_n = 1\})$. We need to prove that $T_{(z,k)} \text{Reg } \mathcal{C} \not\subset (R_0 \times R_0)(\{\eta_n = 0\})$. For this purpose, consider $\alpha = (-R_0 x, R_0 \eta)$. This is an element of $T_z \Omega_0 = J\mathfrak{g}z^\perp$ which satisfies $(R_0 \times R_0)^{-1}(\alpha) \notin \{\eta_n = 0\}$. Moreover, we shall see later in Lemma 7 that $kz = z$ implies $(k - 1)\alpha \in \mathfrak{g}z$ for all $\alpha \in T_z \Omega_0$. Consequently, there exists an $A \in \mathfrak{g}$ such that $(\alpha, Ak) \in T_{(z,k)} \text{Reg } \mathcal{C} \setminus (R_0 \times R_0)(\{\eta_n = 0\})$. The dimension of $\mathcal{C}_{\psi_{\text{wk}}}$ follows from Lemma 4, and the tangent space at (x, η', k) is therefore given by

$$T_{(x,\eta',k)}\mathcal{C}_{\psi_{\text{wk}}} = \{(q, p', Ak) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathfrak{g}k : (R_0(q), R_0(p', 0), Ak) \in T_{(R_0 x, R_0(\eta', 1), k)}(\text{Reg } \mathcal{C})\}. \quad (15)$$

To compute the Hessian of ψ_{wk} at a point $(x_0, \eta'_0, k_0) \in \mathcal{C}_{\psi_{\text{wk}}}$, we fix a basis (A_1, \dots, A_d) of \mathfrak{g} , and use the chart $\alpha : \mathbb{R}^{2n-1} \times \mathbb{R}^d \rightarrow \mathbb{R}^{2n-1} \times G$ defined by

$$\alpha(x, \eta', s) = \left(x, \eta', \exp\left(-\sum_{i=1}^d s_i A_i\right) k_0 \right).$$

With respect to these coordinates, the Hessian of ψ_{wk} is given by the matrix

$$\text{Hess } \psi_{\text{wk}}(x_0, \eta'_0, k_0) = \left(\frac{\partial^2(\psi_{\text{wk}} \circ \alpha)}{\partial X_i \partial X_j}(x_0, \eta'_0, 0) \right)_{1 \leq i, j \leq 2n+d-1}$$

which is a square matrix of size $2n + d - 1$. Before entering the computations, we recall that by (3.17) of [4] we have

$$\langle JA z, B z \rangle_{\mathbb{R}^{2n}} = 0 \quad \forall z \in \Omega_0, \quad \forall A, B \in \mathfrak{g}, \quad (16)$$

which is equivalent to

$$\langle Ax, B\xi \rangle_{\mathbb{R}^n} = \langle Bx, A\xi \rangle_{\mathbb{R}^n} \quad \forall (x, \xi) \in \Omega_0, \quad \forall A, B \in \mathfrak{g}. \quad (17)$$

Using these identities, we obtain for all $(x, \eta', k) \in \mathcal{C}_{\psi_{\text{wk}}}$ that $\text{Hess } \psi_{\text{wk}}(x, \eta', k)$ is given by

$$\left(\begin{array}{c|c|c} 0 & \langle R_0 e_i, (k^{-1} - \mathbf{1}) R_0 e_j \rangle & \langle R_0 e_i, k^{-1} A_j R_0(\eta', 1) \rangle \\ \hline \langle R_0 e_i, (k - \mathbf{1}) R_0 e_j \rangle & 0 & -\langle A_j R_0 x, R_0 e_i \rangle \\ \hline \langle R_0 e_j, k^{-1} A_i R_0(\eta', 1) \rangle & -\langle A_i R_0 x, R_0 e_j \rangle & -\langle A_i R_0 x, A_j R_0(\eta', 1) \rangle \end{array} \right),$$

where the first diagonal block is of size n , the second of size $n - 1$ and the third of size d ; each block has been characterized by specifying the entry of the i th line and the j th column. Let now $(q, p', s) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^d$. We set $A = \sum_{i=1}^d s_i A_i$. Then $(q, p', s) \in \ker \text{Hess } \psi_{\text{wk}}(x, \eta', k)$ if and only if

$$\begin{cases} (\mathbf{1} - k) R_0(p', 0) + A R_0(\eta', 1) = 0, & (a) \\ (k - \mathbf{1}) R_0(q) - A R_0 x = \lambda_0 R_0(e_n), & (b) \\ \langle k R_0(q), A_i R_0(\eta', 1) \rangle - \langle A_i R_0 x, R_0(p', 0) \rangle - \langle A_i R_0 x, A R_0(\eta', 1) \rangle = 0, \quad \forall i = 1, \dots, d, & (c) \end{cases}$$

for some $\lambda_0 \in \mathbb{R}$. Taking the scalar product of (b) with $R_0(\eta', 1)$, we obtain $\lambda_0 = 0$. Using (a), we find that (c) is equivalent to the fact that $\langle kR_0(q), BR_0(\eta', 1) \rangle = \langle kR_0(p', 0), BR_0x \rangle$ for all B in \mathfrak{g} . Since $kR_0x = R_0x$ and $kR_0(\eta', 1) = R_0(\eta', 1)$, we see that for all $B \in \mathfrak{g}$,

$$\langle kR_0(q), BR_0(\eta', 1) \rangle = \langle kR_0(p', 0), BR_0x \rangle \Leftrightarrow (R_0(q), R_0(p', 0)) \in [J\mathfrak{g}(R_0x, R_0(\eta', 1))]^\perp.$$

But then, according to Lemma 4, and Eq. (15), we deduce that

$$\alpha'(\ker \text{Hess } \psi_{\text{wk}}(x, \eta', k)) = T_{(x, \eta', k)} \mathcal{C}_{\psi_{\text{wk}}},$$

which concludes the proof of the proposition. \square

Using the preceding proposition, we are in position to apply Theorem 1 to the integral (14). Nevertheless, since the integrand in (14) also depends on the parameter v , the derivatives of $\tilde{\sigma}_{v\eta_n}(x, \eta)$ with respect to x and η' have to be examined carefully. Indeed, while the derivatives of $\chi_\lambda \circ \Psi_\mu^{-1}$ and $\psi_{\lambda, c} \circ \Psi_\mu^{-1}$ behave nicely in terms of μ , the derivatives of $\tilde{\eta}_c \circ \Psi_\mu^{-1}$ with respect to ξ turn out to be more delicate.

Lemma 5. *For all multiindices α, β , there exists a constant $C > 0$, which depends only on α and β , such that*

$$\sup_{(x, \eta) \in \mathbf{X} \times \mathbb{R}^n} |\partial_x^\beta \partial_{\eta'}^\alpha \tilde{\sigma}_{v\eta_n}(x, \eta)| \leq C \max(1, |v|^{-\delta(|\beta| + |\alpha|)}).$$

Proof. With $\tilde{\sigma}_{v\eta_n}(x, \eta) = \sigma_{v\eta_n}(x, \eta_n(\eta_1, \dots, 1)) = \tau_{v\eta_n}(x, (\eta_1, \dots, 1)/v)$, $\tau_\mu = [(\eta_{\lambda, -2}^2 \chi_\lambda)^2 \times (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda) + f_\lambda]$ one computes

$$\begin{aligned} |\partial_x^\beta \partial_{\eta'}^\alpha \tilde{\sigma}_{v\eta_n}(x, \eta)| &= |v|^{-|\alpha|} |(\partial_x^\beta \partial_{\eta'}^\alpha \tau_{v\eta_n})(x, (\eta', 1)/v)| \\ &\leq C_{\alpha, \beta} |v|^{-|\alpha|} (1 + |x|^2 + (|\eta'|^2 + 1)/v^2)^{(\delta|\beta| - (1-\delta)|\alpha|)/2} \\ &\leq C_{\alpha, \beta} |v|^{-\delta|\alpha|} |v|^{-\delta|\beta|} (v^2 + |vx|^2 + |\eta'|^2 + 1)^{(\delta|\beta| - (1-\delta)|\alpha|)/2} \\ &\leq C_{\alpha, \beta} |v|^{-\delta(|\alpha| + |\beta|)} (v^2 + |vx|^2 + |\eta'|^2 + 1)^{\delta|\beta|/2}. \end{aligned}$$

Since by (38) σ_μ has support in a compact set independent of μ , we obtain an estimate of order $O(1)$ for large v , and one of order $O(v^{-\delta(|\alpha| + |\beta|)})$ for small v . \square

It is interesting to note that similar bounds for $\partial_\xi^\alpha \partial_x^\beta \sigma_\mu$ do not exist; indeed, the fact of considering only differential operators which are transversal to $\text{Reg } \mathcal{C}$ in the variable ξ turns out to be decisive. We can now give an asymptotic expansion for $I(\mu)$.

Theorem 5. *There exists a constant $C > 0$ independent of μ such that for all $\mu > 0$, and all $\delta \in (1/4, 1/2)$,*

$$|I(\mu) - (2\pi\mu)^\kappa L_0(\mu)| \leq C\mu^{\kappa+1-2\delta},$$

where κ is given by Definition 3, and

$$L_0(\mu) = \frac{1}{(2\pi)^n} \int_{0 \leq |\eta_n| \leq E_0} \int_{C_{\psi_{wk}}} \frac{\overline{\chi(k)} \tilde{\sigma}_\mu(x, \eta', \eta_n)}{|\det \psi''_{wk}(x, \eta', k)|_{N(x, \eta', k)} C_{\psi_{wk}}|^{1/2}} d\sigma_{C_{\psi_{wk}}}(x, \eta', k) |\eta_n|^{n-\kappa-1} d\eta_n.$$

Proof. In view of Proposition 3, we can apply Theorem 1 to the integral $I_{\eta_n}(\nu)$ which was defined in (14). Consequently, for each $N \in \mathbb{N}$, there exists a constant $C_N > 0$ independent of η_n such that

$$\left| I_{\eta_n}(\nu) - (2\pi|\nu|)^\kappa \sum_{j=0}^{N-1} |\nu|^j Q_j(\eta_n) \right| \leq C_N |\nu|^N \sup_{|\alpha|+|\beta| \leq 2N} \left\| \partial_{\eta'}^\alpha \partial_x^\beta \tilde{\sigma}_{\nu\eta_n} \right\|_{\infty, \mathbf{X} \times \mathbb{R}^n},$$

as well as constants $\tilde{C}_j > 0$ independent of η_n , such that

$$|Q_j(\eta_n)| \leq \tilde{C}_j \sup_{|\alpha|+|\beta| \leq 2j} \left\| \partial_{\eta'}^\alpha \partial_x^\beta \tilde{\sigma}_{\nu\eta_n} \right\|_{\infty, \mathbf{X} \times \mathbb{R}^n},$$

where, in particular,

$$Q_0(\eta_n) = \int_{C_{\psi_{wk}}} \frac{\overline{\chi(k)} \tilde{\sigma}_{\nu\eta_n}(x, \eta', \eta_n)}{|\det \psi''_{wk}(x, \eta', k)|_{N(x, \eta', k)} C_{\psi_{wk}}|^{1/2}} d\sigma_{C_{\psi_{wk}}}(x, \eta', k).$$

Now, by the previous lemma, for $|\nu| \leq 1$ one has

$$\sup_{|\alpha|+|\beta| \leq 2N} \left\| \partial_{\eta'}^\alpha \partial_x^\beta \tilde{\sigma}_{\nu\eta_n} \right\|_{\infty, \mathbf{X} \times \mathbb{R}^n} \leq c_1 |\nu|^{-2N\delta},$$

where c_1 is some constant depending only on N . Thus, if $|\nu| \leq 1$, we obtain

$$\begin{aligned} & \left| I_{\eta_n}(\nu) - (2\pi|\nu|)^\kappa Q_0(\eta_n) \right| \\ &= \left| I_{\eta_n}(\nu) - (2\pi|\nu|)^\kappa \left[\sum_{j=0}^{N-1} |\nu|^j Q_j(\eta_n) - \sum_{j=1}^{N-1} |\nu|^j Q_j(\eta_n) \right] \right| \\ &\leq C_N |\nu|^N \sup_{|\alpha|+|\beta| \leq 2N} \left\| \partial_{\eta'}^\alpha \partial_x^\beta \tilde{\sigma}_{\nu\eta_n} \right\|_{\infty, \mathbf{X} \times \mathbb{R}^n} + \left| (2\pi|\nu|)^\kappa \sum_{j=1}^{N-1} |\nu|^j Q_j(\eta_n) \right| \\ &\leq c_2 |\nu|^{N(1-2\delta)} + c_3 |\nu|^\kappa \sum_{j=1}^{N-1} |\nu|^{j(1-2\delta)} \end{aligned} \quad (18)$$

with constants $c_i > 0$. Next, let us fix $\varepsilon > 0$, and write

$$I(\mu) = J_1(\mu) + J_2(\mu),$$

where

$$J_1(\mu) = \int_{\varepsilon \leq |\eta_n| \leq E_0} I_{\eta_n}(\mu/\eta_n) |\eta_n|^{n-1} d\eta_n,$$

$$J_2(\mu) = \int_{|\eta_n| \leq \varepsilon} I_{\eta_n}(\mu/\eta_n) |\eta_n|^{n-1} d\eta_n.$$

Since $I_{\eta_n}(\mu)$ is uniformly bounded in η_n and μ ,

$$|J_2(\mu)| \leq c_4 \varepsilon^n, \quad (19)$$

where c_4 is independent of η_n and μ . Now, according to Eq. (18), if $\varepsilon \geq \mu$, then

$$\begin{aligned} & \left| J_1(\mu) - (2\pi\mu)^\kappa \int_{\varepsilon \leq |\eta_n| \leq E_0} Q_0(\eta_n) |\eta_n|^{n-1-\kappa} d\eta_n \right| \\ & \leq C_1 \sum_{j=1}^{N-1} \mu^{\kappa+j(1-2\delta)} \int_{\varepsilon \leq |\eta_n| \leq E_0} |\eta_n|^{n-1-\kappa-j(1-2\delta)} d\eta_n \\ & \quad + C_2 \mu^{N(1-2\delta)} \int_{\varepsilon \leq |\eta_n| \leq E_0} |\eta_n|^{n-1-N(1-2\delta)} d\eta_n \end{aligned}$$

for some constants $C_i > 0$. One easily computes that

$$\begin{aligned} & \left| \int_{\varepsilon \leq |\eta_n| \leq E_0} |\eta_n|^{n-1-\kappa-j(1-2\delta)} d\eta_n \right| \leq C_3 \max\{1, \varepsilon^{n-\kappa-j(1-2\delta)}\}, \\ & \left| \int_{\varepsilon \leq |\eta_n| \leq E_0} |\eta_n|^{n-1-N(1-2\delta)} d\eta_n \right| \leq C_3 \max\{1, \varepsilon^{n-N(1-2\delta)}\}, \end{aligned}$$

so that if we take $\varepsilon = \mu$, which ensures that $|v| \leq 1$ for $J_1(\mu)$, we obtain

$$\begin{aligned} & \left| J_1(\mu) - (2\pi\mu)^\kappa \int_{\mu \leq |\eta_n| \leq E_0} Q_0(\eta_n) |\eta_n|^{n-1-\kappa} d\eta_n \right| \\ & \leq C_1 \max\{\mu^{\kappa+1-2\delta}, \mu^n\} + C_2 \max\{\mu^{N(1-2\delta)}, \mu^n\}. \end{aligned}$$

As the dimension of an orbit of $G \subset O(n)$ in \mathbb{R}^n is at most $n-1$, one necessarily has $\kappa \leq n-1$, yielding $\mu^n = O(\mu^{\kappa+1})$ as μ goes to zero. Therefore, by choosing N large enough, and taking equation (19) together with

$$(2\pi\mu)^\kappa \int_{0 \leq |\eta_n| \leq \mu} Q_0(\eta_n) |\eta_n|^{n-1-\kappa} d\eta_n = O(\mu^n)$$

into account, one gets

$$\left| I(\mu) - (2\pi\mu)^\kappa \int_{0 \leq |\eta_n| \leq E_0} Q_0(\eta_n) |\eta_n|^{n-1-\kappa} d\eta_n \right| \leq C\mu^{\kappa+1-2\delta}.$$

The proof of the theorem is now complete. \square

Remark 3. Note that the strict transform of the critical set \mathcal{C} of ψ is locally given by

$$\tilde{\mathcal{C}}^{\text{st}} = \{(x, \eta, k) \in \mathbb{R}^{2n} \times G : (R_0 x, R_0(\eta', 1), k) \in \text{Reg } \mathcal{C}\} \simeq \mathcal{C}_{\psi_{\text{wk}}} \times \mathbb{R}.$$

The first coefficient in the expansion of Theorem 5 can therefore also be expressed as

$$L_0(\mu) = \frac{1}{(2\pi)^n} \int_{\tilde{\mathcal{C}}^{\text{st}}} \frac{\overline{\chi(k)} \tilde{\sigma}_\mu(x, \eta) |\eta_n|^{n-\kappa-1}}{|\det \psi''_{\text{wk}}(x, \eta', k)_{|N_{(x, \eta', k)} \mathcal{C}_{\text{wk}}}|^{1/2}} d\sigma_{\tilde{\mathcal{C}}}(x, \eta, k). \quad (20)$$

5. Computation of the leading term

In this section, we shall address the question of computing the leading coefficient $L_0(\mu)$ in the expansion of $I(\mu)$. The main result of this section is the following

Proposition 4. *One has*

$$L_0(\mu) = \frac{1}{(2\pi)^n} [\varrho_\chi|_{H_0} : 1] \int_{\text{Reg } \Omega_0} \sigma_\mu(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z}, \quad (21)$$

where $d\sigma_{\text{Reg } \Omega_0}$ is the Riemannian measure on $\text{Reg } \Omega_0$, and $\text{vol } \mathcal{O}_z$ denotes the Riemannian volume of the G -orbit of z . $[\varrho_\chi|_{H_0} : 1]$ stands for the multiplicity of the trivial representation in the restriction of any irreducible representation ϱ_χ corresponding to the character $\chi \in \hat{G}$ to a principal isotropy group H_0 . In particular, the integral on the right-hand side of (21) is convergent.

Note that $\text{Reg } \Omega_0$ is not compact; nevertheless, the existence of the integral in (21) will be deduced on basis of the partial desingularization of \mathcal{C} accomplished in the previous section. Let us start proving Proposition 4, and introduce first certain cut-off functions for $\text{Sing } \Omega_0$.

Definition 4. Let K be compact subset in \mathbb{R}^{2n} as in (38), $\varepsilon > 0$, and denote by v_ε the characteristic function of the set

$$(\text{Sing } \Omega_0 \cap K)_{2\varepsilon} = \{z \in \mathbb{R}^{2n} : |z - z'| < 2\varepsilon \text{ for some } z' \in \text{Sing } \Omega_0 \cap K\}.$$

Consider further the unit ball B_1 in \mathbb{R}^{2n} , and a function $\iota \in C_c^\infty(B_1)$ with $\int \iota dz = 1$, and set $\iota_\varepsilon(z) = \varepsilon^{-2n} \iota(z/\varepsilon)$. Clearly $\int \iota_\varepsilon dz = 1$, $\text{supp } \iota_\varepsilon \subset B_\varepsilon$, and we define

$$u_\varepsilon = v_\varepsilon * \iota_\varepsilon.$$

One can then show that $u_\varepsilon \in C_c^\infty((\text{Sing } \Omega_0 \cap K)_{3\varepsilon})$, and $u_\varepsilon = 1$ on $(\text{Sing } \Omega_0 \cap K)_\varepsilon$, together with

$$|\partial_z^\alpha u_\varepsilon| \leq C_\alpha \varepsilon^{-|\alpha|},$$

where C_α is a constant which depends only on α and n , see Hörmander [15, Theorem 1.4.1].

Next, we shall prove

Lemma 6. *Let $\alpha \in C_c^\infty(\mathbb{R}^{2n})$. Then the limit*

$$\lim_{\varepsilon \rightarrow 0} \int_{\text{Reg } \mathcal{C}} \frac{\overline{\chi(k)}[\alpha(1 - u_\varepsilon)](z)}{|\det \psi''(z, k)|_{N(z, k)} \text{Reg } \mathcal{C}}^{1/2} d\sigma_{\text{Reg } \mathcal{C}}(z, k)$$

exists and is finite. In particular, one has

$$L_0(\mu) = \frac{1}{(2\pi)^n} \lim_{\varepsilon \rightarrow 0} \int_{\text{Reg } \mathcal{C}} \frac{\overline{\chi(k)}[\sigma_\mu(1 - u_\varepsilon)](z)}{|\det \psi''(z, k)|_{N(z, k)} \text{Reg } \mathcal{C}}^{1/2} d\sigma_{\text{Reg } \mathcal{C}}(z, k), \quad (22)$$

where $d\sigma_{\text{Reg } \mathcal{C}}$ is the Riemannian measure on $\text{Reg } \mathcal{C}$.

Proof. With u_ε as in the previous definition, let us define

$$I_\varepsilon(\mu) = \int_G \int_{\mathbf{X}} \int_{\mathbb{R}^n} e^{\frac{i}{\mu} \psi(x, \xi, k)} \overline{\chi(k)}[\alpha(1 - u_\varepsilon)](x, \xi) dx d\xi dk.$$

Since $(x, \xi, k) \in \text{Sing } \mathcal{C}$ implies $(x, \xi) \in \text{Sing } \Omega_0$, a direct application of the generalized theorem of the stationary phase for fixed $\varepsilon > 0$ gives

$$|I_\varepsilon(\mu) - (2\pi\mu)^\kappa L_0(\mu, \varepsilon)| \leq C_\varepsilon \mu^{\kappa+1-2\delta} \quad (23)$$

for some $\delta \in [0, 1/2)$, where $C_\varepsilon > 0$ is a constant depending only on ε , and

$$L_0(\mu, \varepsilon) = \frac{1}{(2\pi)^n} \int_{\text{Reg } \mathcal{C}} \frac{\overline{\chi(k)}[\alpha(1 - u_\varepsilon)](z)}{|\det \psi''(z, k)|_{N(z, k)} \text{Reg } \mathcal{C}}^{1/2} d\sigma_{\text{Reg } \mathcal{C}}(z, k).$$

If α is independent of μ , one has $\delta = 0$. For $\alpha = \sigma_\mu$, the stationary phase theorem has to be applied on $G \times \mathbf{X} \times S^{n-1}$, and $\delta \in (1/4, 1/2)$. On the other hand, applying Theorem 5 to $I_\varepsilon(\mu)$ instead of $I(\mu)$, we obtain again an asymptotic expansion of the form (23) for $I_\varepsilon(\mu)$, where now, according to (20), the first coefficient is given by

$$L_0(\mu, \varepsilon) = \frac{1}{(2\pi)^n} \int_{\tilde{\mathcal{C}}^{\text{st}}} \frac{\overline{\chi(k)}[\alpha(1 - u_\varepsilon) \circ \xi_M \circ \tilde{\varphi}_n](x, \eta) |\eta_n|^{n-\kappa-1}}{|\det \psi''_{\text{wk}}(x, \eta', k)|_{N(x, \eta', k)} \mathcal{C}_{\text{wk}}}^{1/2} d\sigma_{\tilde{\mathcal{C}}}(x, \eta, k).$$

Since the first term in the asymptotic expansion (23) is uniquely determined, the two expressions for $L_0(\mu, \varepsilon)$ must be identical. The statement of the lemma now follows by the Lebesgue theorem on bounded convergence, by which, in particular,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{(2\pi)^n} \int_{\tilde{\mathcal{C}}^{\text{st}}} \frac{\overline{\chi(k)} [\sigma_\mu(1 - u_\varepsilon) \circ \zeta_M \circ \tilde{\varphi}_n](x, \eta) |\eta_n|^{n-\kappa-1}}{|\det \psi''_{\text{wk}}(x, \eta', k)|_{N_{(x, \eta', k)} \mathcal{C}_{\text{wk}}}|^{1/2}} d\sigma_{\tilde{\mathcal{C}}}(x, \eta, k) = L_0(\mu). \quad \square$$

Remark 4. Note that existence of the limit in (22) has been established by partially resolving the singularities of the critical set \mathcal{C} , the corresponding limit being given by the absolutely convergent integral (20).

The proof of the next lemma is mainly algebraic.

Lemma 7. Let α be a smooth, compactly supported function on $\text{Reg } \Omega_0$. Then

$$\int_{\text{Reg } \mathcal{C}} \frac{\overline{\chi(k)} \alpha(z)}{|\det \psi''(z, k)|_{N_{(z, k)} \text{Reg } \mathcal{C}}|^{1/2}} d\sigma_{\text{Reg } \mathcal{C}}(z, k) = [\varrho_{\chi|_{H_0}} : 1] \int_{\text{Reg } \Omega_0} \alpha(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z}.$$

Proof. The main difficulty consists in computing the determinant of the transversal Hessian, which will be accomplished by recurring to previous computations done in [4]. Thus, let (z, k) be a fixed point in $\text{Reg } \mathcal{C}$, and choose an appropriate basis (A_1, \dots, A_d) for \mathfrak{g} as follows. If κ denotes the dimension of \mathcal{O}_z , let

$$\begin{aligned} (A_1, \dots, A_\kappa) &\text{ be an orthonormal basis of } (T_e G_z)^\perp, \\ (A_{\kappa+1}, \dots, A_d) &\text{ be an orthonormal basis of } T_e G_z, \end{aligned}$$

where orthogonality is defined with respect to the scalar product

$$\langle\langle A, B \rangle\rangle = \text{tr}({}^t A B)$$

for arbitrary linear maps A and B in \mathbb{R}^n . From [4] we recall that

$$\det\left(\frac{\psi''(z, k)|_{\mathcal{N}_{(z, k)} \text{Reg } \mathcal{C}}}{i}\right) = \det\left(\frac{\mathcal{A}|_{\mathcal{F}^\perp}}{i}\right),$$

where $\mathcal{A} = \text{Hess } \psi(z, k)$ denotes the Hessian of ψ with respect to the coordinates $(z, s) \rightarrow (z, \exp(\sum_{i=1}^d s_i A_i)k)$, and

$$\mathcal{F} = \left\{ (\alpha, s) \in \mathbb{R}^{2n} \times \mathbb{R}^d : (k - \mathbf{1})\alpha + \sum_{i=1}^d s_i A_i z = 0 \right\}. \quad (24)$$

Next, let (B_1, \dots, B_κ) be in \mathfrak{g} such that $(B_1 z, \dots, B_\kappa z)$ is an orthonormal basis of $\mathfrak{g}z$. For $j = 1, \dots, \kappa$, we define

$$\varepsilon_j = (J B_j z, 0), \quad \varepsilon'_j = ((k^{-1} - \mathbf{1}) B_j z, \langle A_i z, B_j z \rangle, 0) \quad (i = 1, \dots, \kappa). \quad (25)$$

Then $(\varepsilon, \varepsilon')$ constitutes a basis of \mathcal{F}^\perp , see [4, Lemma 3.3]. In what follows, we shall compute $\mathcal{A}|_{\mathcal{F}^\perp}$ in this basis. Writing $\alpha_j = (kB_jx, B_j\xi)$ we find

$$\mathcal{A}\varepsilon_j = ((k^{-1} - \mathbf{1})(\mathbf{1} - \Pi_{\mathfrak{g}z})\alpha_j, 0) + \sum_{r=1}^{\kappa} \langle \alpha_j, B_rz \rangle \varepsilon'_j, \quad (26)$$

where $\Pi_{\mathfrak{g}z}$ is the orthogonal projection onto the space $\mathfrak{g}z$ in \mathbb{R}^{2n} . We state now certain relations that will be crucial for the rest of the computation. For all $(z, k) \in \text{Reg } \mathcal{C}$, we have

$$[k, \Pi_{\mathfrak{g}z}] = 0, \quad [J, k] = 0. \quad (27)$$

$$\text{rank}[(k - \mathbf{1})(\mathbf{1} - \Pi_{\mathfrak{g}z})] \subset J\mathfrak{g}z. \quad (28)$$

The first equality follows easily from the relations $k^{-1}\mathfrak{g}k = \mathfrak{g}$ and $kz = z$, while the second simply says that k is symplectic as a Hamiltonian action in \mathbb{R}^{2n} . In order to establish (28), we differentiate the identity

$$\pi(kz) = \pi(z)$$

with respect to $z \in \Omega_0$, and obtain $(k - \mathbf{1})\alpha \in \ker d_z\pi = \mathfrak{g}z$ for all α in $T_z\Omega_0$, where π denotes the canonical projection of $\mathbb{R}_{(H_0)}^{2n}$ onto the quotient by G . The inclusion (28) now follows by using (12). Coming back to (26), we get

$$\mathcal{A}\varepsilon_j = \sum_{r=1}^{\kappa} -\langle J(k^{-1} - \mathbf{1})\alpha_j, B_rz \rangle \varepsilon_r + \sum_{r=1}^{\kappa} \langle \alpha_j, B_rz \rangle \varepsilon'_r.$$

Using (16), and the fact that $(B_1z, \dots, B_\kappa z)$ is orthonormal, we obtain

$$\mathcal{A}\varepsilon_j = \sum_{r=1}^{\kappa} \langle (\mathbf{1} - k)(\mathbf{1} - k^{-1})B_jx, B_r\xi \rangle \varepsilon_r + \sum_{r=1}^{\kappa} [\langle (k - \mathbf{1})B_jx, B_rx \rangle - \delta_{jr}] \varepsilon'_r, \quad (29)$$

where δ_{jr} is the Kronecker symbol. In the same way we obtain

$$\mathcal{A}\varepsilon'_j = \sum_{r=1}^{\kappa} -\left\langle J(k^{-1} - \mathbf{1})\beta_j + \frac{1}{2}(k^{-1} + I)C_jz, B_rz \right\rangle \varepsilon_r + \sum_{r=1}^{\kappa} \langle \beta_j, B_rz \rangle \varepsilon'_r,$$

where

$$C_j = \sum_{r=1}^{\kappa} \langle A_rz, B_jz \rangle A_r, \quad \beta_j = (k^{-1} - \mathbf{1})(-kB_j\xi, B_jx) - \frac{1}{2}(C_j\xi, C_jx).$$

Let now $f : \mathfrak{g}z \rightarrow \mathfrak{g}z$ be defined by

$$f(\tilde{z}) = \sum_{r=1}^{\kappa} \langle A_rz, \tilde{z} \rangle A_rz, \quad \forall \tilde{z} \in \mathfrak{g}z, \quad (30)$$

and let

$$\Lambda = ((k - \mathbf{1})(k^{-1} - \mathbf{1}) + f)|_{\mathfrak{gz}}$$

be the restriction of the map $(k - \mathbf{1})(k^{-1} - \mathbf{1}) + f$ to \mathfrak{gz} . Note that Λ plays a crucial part in the computations of [4]. Using again (16), one easily gets

$$\mathcal{A}\varepsilon'_j = \sum_{r=1}^{\kappa} \left\langle \begin{pmatrix} k^{-1} & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} \Lambda B_j z, B_r z \right\rangle \varepsilon_r - \sum_{r=1}^{\kappa} \left\langle \begin{pmatrix} 0 & \mathbf{1}_n \\ 0 & 0 \end{pmatrix} \Lambda B_j, B_r z \right\rangle \varepsilon'_r,$$

where the matrices have an obvious meaning. Together with (29), the last equation implies that the matrix of \mathcal{A} in the basis $(\varepsilon, \varepsilon')$ is given by

$$\left(\begin{array}{c|c} \langle (1-k)(1-k^{-1})B_j x, B_i \xi \rangle & \langle \begin{pmatrix} k^{-1} & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} \Lambda B_j z, B_i z \rangle \\ \hline \langle (k-\mathbf{1})B_j x, B_i x \rangle - \delta_{ij} & -\langle \begin{pmatrix} 0 & \mathbf{1}_n \\ 0 & 0 \end{pmatrix} \Lambda B_j, B_i z \rangle \end{array} \right). \quad (31)$$

Let Λ_0 be the matrix of Λ in the basis $(B_1 z, \dots, B_{\kappa} z)$. Then (31) is equal to

$$\left(\begin{array}{c|c} \langle (1-k)(1-k^{-1})B_j x, B_i \xi \rangle & \langle \begin{pmatrix} k^{-1} & 0 \\ 0 & \mathbf{1}_n \end{pmatrix} B_j z, B_i z \rangle \\ \hline \langle (k-\mathbf{1})B_j x, B_i x \rangle - \delta_{ij} & -\langle \begin{pmatrix} 0 & \mathbf{1}_n \\ 0 & 0 \end{pmatrix} B_j z, B_i z \rangle \end{array} \right) \cdot \begin{pmatrix} \mathbf{1}_{\kappa} & 0 \\ 0 & \Lambda_0 \end{pmatrix}.$$

Multiplying by i , and shifting the two columns, we obtain

$$\det \left(\frac{\psi''(z, k)|_{\mathcal{N}(z, k)\mathcal{C}_0}}{i} \right) = \det(\Lambda) \cdot \mathcal{D},$$

where

$$\mathcal{D} = \det \left(\begin{array}{c|c} \langle (k^{-1} - \mathbf{1})B_j x, B_i x \rangle + \delta_{ij} & \langle (k - \mathbf{1})(k^{-1} - \mathbf{1})B_j \xi, B_i x \rangle \\ \hline -\langle B_j \xi, B_i x \rangle & \langle (k - \mathbf{1})B_j x, B_i x \rangle + \delta_{ij} \end{array} \right). \quad (32)$$

We are going to show that $\mathcal{D} = 1$. For this, we introduce the notation

$$U = (B_1 x | \dots | B_{\kappa} x), \quad V = (B_1 \xi | \dots | B_{\kappa} \xi),$$

where $B_j x$ is taken as a column vector in the canonical basis of \mathbb{R}^n . U and V are therefore matrices of size $n \times \kappa$.

Lemma 8. For all $k \in G$ we have

- (a) ${}^t U U + {}^t V V = \mathbf{1}_{\kappa}$;
- (b) ${}^t U V = {}^t V U$;
- (c) k commutes with $U^t U$, $V^t V$, $U^t V$, and $V^t U$;
- (d) $(k - \mathbf{1})U^t V = (k - \mathbf{1})V^t U$;
- (e) $(k - \mathbf{1})(U^t U + V^t V) = k - \mathbf{1}$.

Proof. Part (a) says that $(B_1 z, \dots, B_n z)$ is orthonormal. Part (b) comes from (17). Next, let us denote by X the matrix $X = (B_1 z | \dots | B_n z)$. Then $X^t X$ is the matrix of $\Pi_{\mathfrak{g}z}$ in the canonical basis of \mathbb{R}^{2n} . Moreover,

$$X^t X = \left(\begin{array}{c|c} U^t U & U^t V \\ \hline V^t U & V^t V \end{array} \right).$$

Therefore the property $[\Pi_{\mathfrak{g}z}, k] = 0$, see (27), is equivalent to (c). The two last properties are more subtle. One has to note that (27) is equivalent to

$$\Pi_{\mathfrak{g}z}(k - \mathbf{1})J(\mathbf{1} - \Pi_{\mathfrak{g}z}) = (k - \mathbf{1})J(\mathbf{1} - \Pi_{\mathfrak{g}z}).$$

By expressing this in terms of matrices, one easily obtains (d) and (e). \square

Coming back to the proof of Lemma 7, we rewrite Eq. (32) as

$$\mathcal{D} = \det \left(\begin{array}{c|c} {}^t U(k - \mathbf{1})U + \mathbf{1}_\kappa & {}^t V(k^{-1} - \mathbf{1})(k - \mathbf{1})U \\ \hline -{}^t V U & {}^t U(k^{-1} - \mathbf{1})U + \mathbf{1}_\kappa \end{array} \right) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where we replaced k^{-1} by k . We claim that the blocks c and d commute. Indeed,

$$cd = -{}^t V U {}^t U (k^{-1} - \mathbf{1})U - {}^t V U,$$

$$dc = -{}^t U (k^{-1} - \mathbf{1})U {}^t V U - {}^t V U = -{}^t U (k^{-1} - \mathbf{1})V {}^t U U - {}^t V U,$$

by (d) of Lemma 8. By (c) of Lemma 8, $(k^{-1} - \mathbf{1})$ commutes with $V {}^t U$, and since ${}^t U V = {}^t V U$, by (b), we get $[c, d] = 0$. Therefore, $\mathcal{D} = \det(ad - bc)$. Using (a) and (d) of Lemma 8, it is then a straightforward computation to show that in fact, $ad - bd = \mathbf{1}_\kappa$, yielding $\mathcal{D} = 1$. We have thus shown the equality

$$\det \left(\frac{\psi''(z, k)|_{\mathcal{N}_{(z, k)C_0}}}{i} \right) = \det((k - \mathbf{1})(k^{-1} - \mathbf{1})|_{\mathfrak{g}z} + f),$$

where the map $f : \mathfrak{g}z \rightarrow \mathfrak{g}z$ was defined in (30). The rest of the proof of Lemma 7 now follows by the argument given in [4, Section 3.3.2]. \square

To finish proving Proposition 4, we note that, as a consequence of Lemmata 6 and 7, the limit

$$\lim_{\varepsilon \rightarrow 0} \int_{\text{Reg } \Omega_0} [\alpha(1 - u_\varepsilon)](z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z}$$

exists for any $\alpha \in C_c^\infty(\mathbb{R}^{2n})$ and is finite. Assume now that α is non-negative. Since $|u_\varepsilon| \leq 1$, the Lemma of Fatou implies

$$\int_{\text{Reg } \Omega_0} \lim_{\varepsilon \rightarrow 0} [\alpha(1 - u_\varepsilon)](z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z} \leq \lim_{\varepsilon \rightarrow 0} \int_{\text{Reg } \Omega_0} [\alpha(1 - u_\varepsilon)](z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z} < \infty,$$

which means that

$$\int_{\text{Reg } \Omega_0} \alpha(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z} < \infty \quad \forall \alpha \in C_c^\infty(\mathbb{R}^{2n}, \mathbb{R}_+). \quad (33)$$

In particular, if α is taken to be equal 1 on the compact set K specified in (38), we obtain

$$\int_{\text{Reg } \Omega_0} |\sigma_\mu(z)| \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z} \leq C \int_{\text{Reg } \Omega_0} \alpha(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z} < \infty \quad (34)$$

for some $C > 0$. Now, by Lemmata 6 and 7,

$$L_0(\mu) = \frac{1}{(2\pi)^n} [\varrho_{\chi|H_0} : 1] \lim_{\varepsilon \rightarrow 0} \int_{\text{Reg } \Omega_0} [\sigma_\mu(1 - u_\varepsilon)](z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z}. \quad (35)$$

Since (34) implies that the integrand in (35) has an integrable majorant for arbitrary ε , we can apply the Lebesgue theorem of bounded convergence to obtain

$$L_0(\mu) = \frac{1}{(2\pi)^n} [\varrho_{\chi|H_0} : 1] \int_{\text{Reg } \Omega_0} \sigma_\mu(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z}.$$

This completes the proof of Proposition 4. \square

So far we have shown that $\text{tr } P_\chi \mathcal{E}_\lambda = d_\chi \lambda^{n/2m} I(\lambda^{-1/2m}) + O(1)$, where

$$I(\mu) = \frac{\mu^\kappa}{(2\pi)^{n-\kappa}} [\varrho_{\chi|H_0} : 1] \int_{\text{Reg } \Omega_0} \sigma_\mu(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{Vol } \mathcal{O}_z} + O(\mu^{\kappa+1-2\delta}), \quad (36)$$

$\delta \in (1/4, 1/2)$, and $\sigma_\mu = [(\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda) + f_\lambda] \circ \Psi_\mu^{-1}$ with $\lambda = \mu^{-2m}$. In particular, the last integral exists, and is finite, so that in order to finish the computation of the leading term in the asymptotic expansion for $\text{tr } P_\chi \mathcal{E}_\lambda$, we are left with the task of examining the latter integral. To characterize the support of σ_μ , let us introduce the sets

$$W_\lambda = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : a_\lambda < 0\},$$

$$A_{c, \lambda} = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : a_\lambda < c(h^{\delta-\omega} + d)\}, \quad B_{c, \lambda} = \mathbf{X} \times \mathbb{R}^n - A_{c, \lambda},$$

$$D_c = (\partial \mathbf{X} \times \mathbb{R}^n)(c, h^{-2\delta} g),$$

$$F_\lambda = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : \chi_\lambda = 0 \text{ or } \eta_{\lambda, -2} = 0 \text{ or } \chi_\lambda = \eta_{\lambda, -2} = 1\},$$

$$\mathcal{RV}_{c, \lambda} = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : |a_\lambda| < c(h^{\delta-\omega} + d)\} \cup \{(x, \xi) \in D_c : x \in \mathbf{X}, a_\lambda < c(h^{\delta-\omega} + d)\}.$$

Note that

$$D_c = \{(x, \xi) \in \mathbb{R}^{2n} : \text{dist}(x, \partial \mathbf{X}) < \sqrt{c}(1 + |x|^2 + |\xi|^2)^{-\delta/2}\},$$

since for

$$h^{-2\delta}(x, \xi)g_{(x, \xi)}(x - y, \xi - \eta) = (1 + |x|^2 + |\xi|^2)^\delta \left[\frac{|\xi - \eta|^2}{1 + |x|^2 + |\xi|^2} + |x - y|^2 \right] < c$$

to hold for some $(y, \eta) \in \partial \mathbf{X} \times \mathbb{R}^n$, it is necessary and sufficient that $|x - y|^2(1 + |x|^2 + |\xi|^2)^\delta < c$ is satisfied for some $y \in \partial \mathbf{X}$.

Lemma 9. *For sufficiently large $c > 0$ one has:*

- (i) $\text{supp } f_\lambda \subset \mathcal{RV}_{c, \lambda} \subset A_{c, \lambda}$;
- (ii) $\text{supp}(\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda) \subset A_{c, \lambda}$;
- (iii) $(\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda) = 1$ on $W_\lambda \cap \mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} \mathcal{RV}_{c, \lambda}$.

Proof. As already explained in Part I, Eq. (51), the support of f_λ is contained in $\mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} F_\lambda$, the complement of F_λ in $\mathbf{X} \times \mathbb{R}^n$. Furthermore, for sufficiently large $c > 0$, the set $\mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} F_\lambda$ is contained in $\mathcal{RV}_{c, \lambda}$, which is a consequence of the inclusions

$$\mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} F_\lambda \subset A_{c, \lambda} \cap \mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} E_\lambda \subset \mathcal{RV}_{c, \lambda}, \quad (37)$$

where $E_\lambda = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : (x, \xi) \notin D_4, a_\lambda < -4h^{\delta-\omega} - 8C_0d\}$, see Part I, Lemma 16. Next, we note that $(\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda)(x, \xi)$ must be equal 1 on $W_\lambda \cap \mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} \mathcal{RV}_{c, \lambda}$, since according to (37) we have the inclusion $\mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} \mathcal{RV}_{c, \lambda} \subset B_{c, \lambda} \cup E_\lambda$, and hence $W_\lambda \cap \mathbb{G}_{\mathbf{X} \times \mathbb{R}^n} \mathcal{RV}_{c, \lambda} \subset E_\lambda \subset \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : \chi_\lambda = \eta_{\lambda, -2} = 1\}$, due to the fact that $W_\lambda \cap B_{c, \lambda} = \emptyset$. Furthermore, $(\eta_{\lambda, -2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda, -2}^2 \chi_\lambda)(x, \xi)$ vanishes on $B_{c, \lambda}$, since for large c , $(x, \xi) \in B_{c, \lambda}$ implies $(x, \xi) \notin \mathcal{M}_\lambda(1, h^{-2\delta}g)$, by the proof of the previous lemma. \square

Consequently, by introducing the sets

$$\begin{aligned} \widetilde{W}_\mu &= \Psi_\mu(W_{\mu^{-2m}}) = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : b_{\mu^{-2m}} < 0\}, \\ \widetilde{A}_{c, \mu} &= \Psi_\mu(A_{c, \mu^{-2m}}) = \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : b_{\mu^{-2m}} < c(h^{\delta-\omega} + d) \circ \Psi_\mu^{-1}\}, \\ \widetilde{B}_{c, \mu} &= \mathbf{X} \times \mathbb{R}^n - \widetilde{A}_{c, \mu}, \\ \widetilde{\mathcal{RV}}_{c, \mu} &= \Psi_\mu(\mathcal{RV}_{c, \mu^{-2m}}) \\ &= \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : |b_{\mu^{-2m}}| < c(h^{\delta-\omega} + d) \circ \Psi_\mu^{-1}\} \\ &\quad \cup \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : (x, \xi/\mu) \in D_c, b_{\mu^{-2m}} < c(h^{\delta-\omega} + d) \circ \Psi_\mu^{-1}\}, \end{aligned}$$

one sees that, for all $\mu \in \mathbb{R}_*^+$,

$$\text{supp } \sigma_\mu \subset \widetilde{A}_{c, \mu} \subset K \quad (38)$$

for some sufficiently large $c > 0$, and some suitable compact subset $K \subset \mathbb{R}^{2n}$. We proceed now to split the integral in (36) into the three integrals

$$\int_{\text{Reg } \Omega_0 \cap \widetilde{W}_\mu} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} - \int_{\text{Reg } \Omega_0 \cap \widetilde{W}_\mu \cap \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} + \int_{\text{Reg } \Omega_0 \cap \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}} \sigma_\mu(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z}, \quad (39)$$

where we made use of the fact that, since $W_\lambda, \mathcal{R}\mathcal{V}_{c,\lambda}$ are contained in $A_{c,\lambda}$, and $\mathcal{C}_{A_{c,\lambda}} \mathcal{R}\mathcal{V}_{c,\lambda} \subset W_\lambda$, one has $A_{c,\lambda} - W_\lambda \cap \mathcal{C}_{\mathbf{X} \times \mathbb{R}^n} \mathcal{R}\mathcal{V}_{c,\lambda} = \mathcal{R}\mathcal{V}_{c,\lambda}$. The next lemma will show that the main contribution to $L_0(\mu)$ is actually given by the first integral in (39), provided that we make the following assumption.

Assumption 2. There exists a constant $c > 0$ such that for sufficiently small $\varrho > 0$, $\text{vol}(\partial \mathbf{X})_\varrho \leq c\varrho$. Furthermore, $0 \notin \partial \mathbf{X}$.

Lemma 10. *Put*

$$\begin{aligned} \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(1)} &= \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : |b| < c(h^{\delta-\omega} + d) \circ \Psi_\mu^{-1}\}, \\ \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(2)} &= \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : (x, \xi/\mu) \in D_c, b < c(h^{\delta-\omega} + d) \circ \Psi_\mu^{-1}\}, \end{aligned}$$

so that $\widetilde{\mathcal{R}\mathcal{V}}_{c,\mu} = \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(1)} \cup \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(2)}$. Then, as $\mu \rightarrow 0$,

$$\begin{aligned} \int_{\text{Reg } \Omega_0 \cap \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(1)}} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} &= O(\mu^{2\delta-\frac{1}{2}}), \\ \int_{\text{Reg } \Omega_0 \cap \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(2)}} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} &= O(\mu^{\frac{\delta}{1+\delta}}), \end{aligned}$$

for arbitrary $\delta \in (1/4, 1/2)$.

Proof. Let $\mathbf{1}_A$ denote the characteristic function of the set A . As already noted, Ω_0 is homogeneous in x and ξ , meaning that $(x, \xi) \in \Omega_0$ implies $(sx, t\xi) \in \Omega_0$ for all $s, t \in \mathbb{R}$. Furthermore, by Lemma 3, $\{(x, \xi) \in \text{Reg } \Omega_0 : \xi \in \text{Sing } \mathbb{R}^n\}$ is a subset of measure zero in $\text{Reg } \Omega_0$. Consequently, we can parametrize $\text{Reg } \Omega_0$ up to a set of measure zero as follows. Take $z = (x, \xi) \in \Omega_0$, $\xi \in \mathbb{R}_{(H_0)}^n$, and let $\xi = s\eta$, $x = r\vartheta$ be polar coordinates in \mathbb{R}^n , and $N_\xi \mathcal{O}_\xi$, respectively, where $r, s > 0$, and $\eta \in S^{n-1}$, $\vartheta \in S^{n-\kappa-1}$. In this coordinates one computes then

$$\begin{aligned} &\int_{\text{Reg } \Omega_0 \cap \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} \\ &= \int_{\mathbb{R}_{(H_0)}^n} \left(\int_{N_\xi \mathcal{O}_\xi} \mathbf{1}_{\widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}}(x, \xi) \frac{d\sigma_{N_\xi \mathcal{O}_\xi}(x)}{\text{vol } \mathcal{O}_{(x,\xi)}} \right) d\xi \\ &= \int_0^\infty \int_{S_{(H_0)}^{n-1}} \left(\int_0^\infty \int_{N_{s\eta}^1 \mathcal{O}_{s\eta}} \mathbf{1}_{\widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}}(r\vartheta, s\eta) s^{n-1} r^{n-\kappa-1} \frac{dr d\vartheta}{\text{vol } \mathcal{O}_{(r\vartheta, s\eta)}} \right) ds d\eta, \quad (40) \end{aligned}$$

since

$$\sqrt{\det g|_{\text{Reg } \Omega_0}}(r, s, \vartheta, \eta) = s^{n-1} r^{n-\kappa-1} d\vartheta d\eta,$$

where $g|_{\text{Reg } \Omega_0}$ denotes the induced metric on $\text{Reg } \Omega_0$, and $d\eta$ and $d\vartheta$ are the volume elements of S^{n-1} and $N_x^1 \mathcal{O}_x = \{v \in N_x \mathcal{O}_x : \|v\| = 1\}$, respectively. Now, the condition $b(x, \xi) < c(h^{\delta-\omega} + d)(x, \xi/\mu)$ implies that $|\xi| < c_1$, see Part I, Eq. (60); here, and in what follows, $c_i > 0$ will denote positive constants. Hence,

$$\begin{aligned} \widetilde{\mathcal{RV}}_{c,\mu}^{(2)} &\subset \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : c_0 \mu^{\varepsilon_2} \leq |\xi| \leq c_1, \text{dist}(x, \partial \mathbf{X}) < c_2 |\xi|^{-\delta} \mu^\delta\} \\ &\cup \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : |\xi| < c_0 \mu^{\varepsilon_2}\} \\ &\subset [(\partial \mathbf{X})_{c_3 \mu^{\delta(1-\varepsilon_2)}} \times B^n(c_1)] \cup [\mathbf{X} \times B^n(c_0 \mu^{\varepsilon_2})], \end{aligned}$$

where $B^n(\varrho)$ denotes the ball of radius ϱ in n -dimensional Euclidean space, and $1 > \varepsilon_2 > 0$ will be chosen later. On the other hand, the proof of Lemma 18 in Part I implies that, for small μ , and some $0 < \varepsilon_1 < 1$ to be specified later,

$$\widetilde{\mathcal{RV}}_{c,\mu}^{(1)} \subset \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : c_4 \leq |\xi| \leq c_1, |1 - 1/a_{2m}(x, \xi)| \leq c_5 \mu^{\delta-\omega}\} \cup [\mathbf{X} \times B^n(\mu^{\varepsilon_1})].$$

Now, using the parametrization of $\text{Reg } \Omega_0$ specified above, one sees that for small $\varrho > 0$

$$\int_{\text{Reg } \Omega_0 \cap [\mathbf{X} \times B^n(\varrho)]} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} = \int_0^\varrho \int_{S_{(H_0)}^{n-1}} \left(\int_{N_{s\eta} \mathcal{O}_{s\eta}} \mathbf{1}_{\mathbf{X}}(x) \frac{d\sigma_{N_{s\eta} \mathcal{O}_{s\eta}}(x)}{\text{vol } \mathcal{O}_{(x,s\eta)}} \right) s^{n-1} ds d\eta = O(\varrho),$$

where we took into account that $\text{vol } \mathcal{O}_{(x,s\eta)}$ is at most of order s^κ for small s , and $\kappa \leq n-1$. Therefore, the restriction of the integral (40) to $\text{Reg } \Omega_0 \cap \widetilde{\mathcal{RV}}_{c,\mu}^{(1)}$ can be estimated from above by

$$\int_{\text{Reg } \Omega_0} \mathbf{1}_{\{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : c_4 \leq |\xi| \leq c_1, |1-1/a_{2m}(x,\xi)| \leq c_5 \mu^{\delta-\omega}\}}(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} + O(\mu^{\varepsilon_1}).$$

Now, by letting $x \in \mathbb{R}_{(H_0)}^n$, $\xi \in N_x \mathcal{O}_x$, and interchanging the roles of x and ξ , we obtain

$$\begin{aligned} &\int_{\mathbf{X} \cap \mathbb{R}_{(H_0)}^n} \left(\int_{c_4}^{c_1} \int_{N_x^1 \mathcal{O}_x} \mathbf{1}_{\{(x',\xi) : |1-1/a_{2m}(x',\xi)| \leq c_5 \mu^{\delta-\omega}\}}(x, s\eta) \frac{s^{n-\kappa-1} ds d\eta}{\text{vol } \mathcal{O}_{(x,s\eta)}} \right) dx \\ &= \int_{\mathbf{X} \cap \mathbb{R}_{(H_0)}^n} \left(\int_{\{\varsigma : |\varsigma-1| \leq c_5 \mu^{\delta-\omega}\}} \int_{N_x^1 \mathcal{O}_x} \varsigma^{-1} \left(\frac{1}{\varsigma a_{2m}(x, \eta)} \right)^{\frac{n-\kappa}{2m}} \frac{\mathbf{1}_{[c_1, c_4]}((\varsigma a_{2m}(x, \eta))^{-\frac{1}{2m}}) d\varsigma d\eta}{\text{vol } \mathcal{O}_{(x, (\varsigma a_{2m}(x, \eta))^{-1/2m} \eta)}} \right) dx \\ &\leq c_6 \int_{\{\varsigma : |\varsigma-1| \leq c_5 \mu^{\delta-\omega}\}} d\varsigma = O(\mu^{\delta-\omega}), \end{aligned}$$

where we made the change of variables $\varsigma = |\xi|^{-2m}/a_{2m}(x, \xi/|\xi|) = s^{-2m}/a_{2m}(x, \eta)$, and used the fact that $(1+z)^\beta - (1-z)^\beta = O(|z|)$ for arbitrary $z \in \mathbb{C}$, $|z| < 1$, and $\beta \in \mathbb{R}$. Note that due to the ellipticity condition (3), $a_{2m}(x, \eta) \geq C_0 > 0$ for all $\eta \in N_x^1 \mathcal{O}_x$ and $x \in \mathbf{X}$. Putting $\varepsilon_1 = \delta - \omega = 2\delta - 1/2$ therefore yields

$$\int_{\text{Reg } \Omega_0 \cap \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(1)}} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} = O(\mu^{2\delta-\frac{1}{2}}).$$

Let us now assume that Assumption 2 is fulfilled. Then, for small μ , the restriction of the integral (40) to $\text{Reg } \Omega_0 \cap \widetilde{\mathcal{R}\mathcal{V}}_{c,\mu}^{(2)}$ can be estimated from above by

$$\begin{aligned} & \int_{\text{Reg } \Omega_0} \mathbf{1}_{[(\partial \mathbf{X})_{c_3\mu^{\delta(1-\varepsilon_2)} \times B^n(c_1)]}(z) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} + O(\mu^{\varepsilon_2}) \\ &= \int_{\mathbb{R}_{(H_0)}^n \cap (\partial \mathbf{X})_{c_3\mu^{\delta(1-\varepsilon_2)}}} \left(\int_{N_x \mathcal{O}_x \cap B^n(c_1)} \frac{d\sigma_{N_x \mathcal{O}_x}(\xi)}{\text{vol } \mathcal{O}_{(x,\xi)}} \right) dx + O(\mu^{\varepsilon_2}) \\ &\leq c_7 \text{vol}(\partial \mathbf{X})_{c_3\mu^{\delta(1-\varepsilon_2)}} + O(\mu^{\varepsilon_2}) = O(\mu^{\delta(1-\varepsilon_2)}) + O(\mu^{\varepsilon_2}) = O(\mu^{\frac{\delta}{1+\delta}}), \end{aligned}$$

where we put $\varepsilon_2 = \delta/(1+\delta)$, and took into account that, since $0 \notin \partial \mathbf{X}$, the integrand of the last integral over x is bounded on $\mathbb{R}_{(H_0)}^n \cap (\partial \mathbf{X})_{c_3\mu^{\delta(1-\varepsilon_2)}}$ by some constant independent of μ . The assertion of the lemma now follows. \square

Now, for $x \in \mathbf{X}$, $|\xi| > \mu$, the condition $b_{\mu^{-2m}}(x, \xi) < 0$ is equivalent to $a_{2m}(x, \xi) < 1$, due to the ellipticity condition (3), and Eq. (7). By using arguments similar to those given in the proof of the previous lemma one therefore computes

$$\begin{aligned} & \int_{\text{Reg } \Omega_0 \cap \widetilde{W}_\mu} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} \leq \int_{\text{Reg } \Omega_0 \cap [\mathbf{X} \times B^n(\mu)]} \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} + \int_{\text{Reg } \Omega_0} \mathbf{1}_{(-\infty, 1]}(a_{2m}(z)) \frac{d\sigma_{\text{Reg } \Omega_0}(z)}{\text{vol } \mathcal{O}_z} \\ &= O(\mu) + \int_{\text{Reg } \Omega_0/G} \mathbf{1}_{(-\infty, 1]}(a_{2m}[z]) d\sigma_{\text{Reg } \Omega_0/G}([z]) \\ &= O(\mu) + \text{vol}([a_{2m}^{-1}((-\infty, 1]) \cap \text{Reg } \Omega_0]/G), \end{aligned} \quad (41)$$

where we took into account [4, Eq. (3.37)]. Here the latter volume is defined in the sense of [10, Section 3.H.2]. This finishes the computation of the leading term. Collecting everything together, we obtain

Proposition 5. *As $\lambda \rightarrow +\infty$, one has*

$$\left| \text{tr } P_X \mathcal{E}_\lambda - \frac{d_X[\varrho_X|_{H_0} : 1]}{(2\pi)^{n-\kappa}} \text{vol}([a_{2m}^{-1}((-\infty, 1]) \cap \Omega_0]/G) \lambda^{(n-\kappa)/2m} \right| = O(\lambda^{(n-\kappa-1/4)/2m}),$$

Furthermore, a similar result holds for the trace of $(P_X \mathcal{E}_\lambda)^2$, too.

Proof. Since $\text{tr } P_\chi \mathcal{E}_\lambda = d_\chi \lambda^{n/2m} I(\lambda^{-1/2m}) + O(1)$, the assertion follows with Theorem 5 and Proposition 4, together with Eqs. (39), (41), and Lemma 10, by taking into account that

$$\max_{\delta \in (1/4, 1/2)} \min \left(\frac{\delta}{1+\delta}, 1-2\delta, 2\delta - \frac{1}{2} \right) = \frac{1}{4}.$$

Finally, if in all the previous computations \mathcal{E}_λ is replaced by \mathcal{E}_λ^2 , we obtain a similar estimate for the trace of $P_\chi \mathcal{E}_\lambda \cdot P_\chi \mathcal{E}_\lambda = P_\chi \mathcal{E}_\lambda^2$. \square

6. Proof of the main result

As a consequence of Lemma 11 of Part I, and Proposition 5, we get the following.

Theorem 6. Let $N_\chi^{\mathcal{E}_\lambda}$ be the number of eigenvalues of \mathcal{E}_λ which are $\geq 1/2$ and whose eigenfunctions are contained in the χ -isotypic component \mathcal{H}_χ of $L^2(\mathbb{R}^n)$, and assume that Assumptions 1 and 2 are satisfied. Then

$$\left| N_\chi^{\mathcal{E}_\lambda} - \frac{d_\chi [\varrho_\chi|_{H_0} : 1]}{(2\pi)^{n-\kappa}} \text{vol}([a_{2m}^{-1}((-\infty, 1]) \cap \Omega_0]/G) \lambda^{(n-\kappa)/2m} \right| = O(\lambda^{(n-\kappa-1/4)/2m}),$$

as $\lambda \rightarrow +\infty$.

Similar estimates for the traces of $\tilde{\mathcal{F}}_\lambda$ and \mathcal{F}_λ can be derived as well, and using Lemma 12 of Part I we obtain the following.

Theorem 7. Let $M_\chi^{\mathcal{F}_\lambda}$ be the number of eigenvalues of \mathcal{F}_λ which are $\geq 1/2$ and whose eigenfunctions are contained in the χ -isotypic component \mathcal{H}_χ of $L^2(\mathbb{R}^n)$. Under Assumptions 1 and 2 one has then

$$\left| M_\chi^{\mathcal{F}_\lambda} - \frac{d_\chi [\varrho_\chi|_{H_0} : 1]}{(2\pi)^{n-\kappa}} \text{vol}([a_{2m}^{-1}((-\infty, 1]) \cap \Omega_0]/G) \lambda^{(n-\kappa)/2m} \right| = O(\lambda^{(n-\kappa-1/4)/2m}),$$

as $\lambda \rightarrow +\infty$.

Proof. The proof is similar to the one of Theorem 6; in analogy to Eq. (9) one has

$$\sigma^l(\mathcal{F}_\lambda) = (\eta_2^2 \chi_\lambda^+)^2 (3 - 2\eta_2^2 \chi_\lambda^+) + f_\lambda + r_\lambda,$$

where $r_\lambda \in S^{-\infty}(h^{-2\delta}g, 1)$, and $f_\lambda \in S(h^{-2\delta}g, h^{1-2\delta})$, everything uniformly in λ . Again we have $\text{supp } f_\lambda \subset \mathcal{R}\mathcal{V}_{c,\lambda}$ for sufficiently large c , and $\iint |r_\lambda(x, \xi)| dx d\xi \leq C$ for some constant $C > 0$ independent of λ , so that in order to study the asymptotic behavior of $\text{tr } P_\chi \mathcal{F}_\lambda$, we can restrict ourselves to the integral

$$\int_G \iint \overline{\chi(k)} e^{i(x-kx)\xi} ((\eta_2^2 \chi_\lambda^+)^2 (3 - 2\eta_2^2 \chi_\lambda^+) + f_\lambda)(x, \xi) dx d\xi dk.$$

An application of the method of the stationary phase then yields the desired result. \square

We are now in position to prove our main result. In the case $G = \{\mathbf{1}\}$, one has $\Omega_0 = \mathbb{R}^{2n}$, and we simply obtain [18, Theorem 13.1].

Theorem 8. *Let G be a compact group of isometries in Euclidean space \mathbb{R}^n , H_0 a principal isotropy group, and $\mathbf{X} \subset \mathbb{R}^n$ a bounded open set invariant under G . Assume that:*

- (i) *for sufficiently small $\varrho > 0$, $\text{vol}(\partial\mathbf{X})_{\varrho} \leq c\varrho$, where $c > 0$ is a constant independent of ϱ , and $0 \notin \partial\mathbf{X}$;*
- (ii) *the set $\text{Sing } \mathbb{R}^n = \mathbb{R}^n \setminus \mathbb{R}_{(H_0)}^n$ is included in a strict vector subspace F of \mathbb{R}^n of dimension $r < n$.*

Let further A_0 be a symmetric, classical pseudodifferential operator in $L^2(\mathbb{R}^n)$ of order $2m$ with principal symbol a_{2m} that commutes with the regular representation of G in $L^2(\mathbb{R}^n)$, and assume that A_0 satisfies the ellipticity condition (3). Consider further the Friedrichs extension of the operator

$$\text{res} \circ A_0 \circ \text{ext} : C_c^\infty(\mathbf{X}) \rightarrow L^2(\mathbf{X}),$$

and denote it by A . Then A has discrete spectrum. Furthermore, if $N_\chi(\lambda)$ denotes the number of eigenvalues of A less or equal λ and with eigenfunctions in the χ -isotypic component $\text{res } \mathcal{H}_\chi$ of $L^2(\mathbf{X})$, and $\kappa = \dim H_0$, then

$$N_\chi(\lambda) = \frac{d_\chi [\varrho_\chi|_{H_0} : 1]}{(2\pi)^{n-\kappa}} \text{vol}([a_{2m}^{-1}((-\infty, 1]) \cap \Omega_0]/G) \lambda^{(n-\kappa)/2m} + O(\lambda^{(n-\kappa-1/4)/2m}),$$

where d_χ denotes the dimension of any unitary irreducible representation ϱ_χ corresponding to the character χ , and $[\varrho_\chi|_{H_0} : 1]$ is the multiplicity of the trivial representation in the restriction of ϱ_χ to H_0 .

Proof. The discreteness of the spectrum was already shown in Proposition 1. Now, by Theorems 4 and 5 of Part I, there exist constants $C_i > 0$ independent of λ such that

$$N_\chi^{\mathcal{E}_\lambda} - C_1 \leq \mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap C_c^\infty(\mathbf{X})) \leq M_\chi^{\mathcal{F}_\lambda} + C_2.$$

Theorems 6 and 7 then yield the estimate

$$\left| N_\chi(\lambda) - \frac{d_\chi [\varrho_\chi|_{H_0} : 1]}{(2\pi)^{n-\kappa}} \text{vol}([a_{2m}^{-1}((-\infty, 1]) \cap \Omega_0]/G) \lambda^{(n-\kappa)/2m} \right| = O(\lambda^{(n-\kappa-1/4)/2m}). \quad \square$$

Remark 5. Note that if G is a finite group, we recover exactly the first term in the asymptotic expansion for $N_\chi(\lambda)$ given in Part I, since in this case $\Omega_0 = \mathbb{R}^{2n}$, $H_0 = \{\mathbf{1}\}$, $[\varrho_\chi|_{H_0} : 1] = d_\chi$, and $\kappa = 0$, while $\text{vol}(a_{2m}^{-1}((-\infty, 1])/G) = \text{vol}(a_{2m}^{-1}((-\infty, 1]))/|G|$.

References

- [1] G.E. Bredon, Introduction to Compact Transformation Groups, Pure Appl. Math., vol. 46, Academic Press, New York, 1972.

- [2] M. Bronstein, V. Ivrii, Sharp spectral asymptotics for operators with irregular coefficients I. Pushing the limits, *Comm. Partial Differential Equations* 28 (2003) 83–102.
- [3] J. Brüning, E. Heintze, Representations of compact Lie groups and elliptic operators, *Invent. Math.* 50 (1979) 169–203.
- [4] R. Cassanas, Reduced Gutzwiller formula with symmetry: Case of a Lie group, *J. Math. Pures Appl.* 85 (2006) 719–742.
- [5] M. Combes, J. Ralston, D. Robert, A proof of the Gutzwiller semiclassical trace formula using coherent states decomposition, *Comm. Math. Phys.* 202 (1999) 463–480.
- [6] H. Donnelly, G-spaces, the asymptotic splitting of $L^2(M)$ into irreducibles, *Math. Ann.* 237 (1978) 23–40.
- [7] Z. El Houakmi, B. Helffer, Comportement semi-classique en présence de symétries: action d'un groupe de Lie compact, *Asymptotic Anal.* 5 (2) (1991) 91–113.
- [8] C. Emmrich, H. Römer, Orbifolds as configuration spaces of systems with gauge symmetries, *Comm. Math. Phys.* 129 (1) (1990) 69–94.
- [9] V.I. Feigin, Asymptotic distribution of eigenvalues for hypoelliptic systems in \mathbb{R}^n , *Math. USSR Sb.* 28 (4) (1976) 533–552.
- [10] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry*, third ed., Springer-Verlag, Berlin, 2004.
- [11] A. Grigis, J. Sjöstrand, *Microlocal Analysis for Differential Operators*, London Math. Soc. Lecture Note Ser., vol. 196, Cambridge Univ. Press, Cambridge, 1994.
- [12] B. Helffer, D. Robert, Etude du spectre pour un opérateur globalement elliptique dont le symbole de Weyl présente des symétries II, *Amer. J. Math.* 108 (1986) 973–1000.
- [13] L. Hörmander, The spectral function of an elliptic operator, *Acta Math.* 121 (1968) 193–218.
- [14] L. Hörmander, The Weyl calculus of pseudo-differential operators, *Comm. Pure Appl. Math.* 32 (1979) 359–443.
- [15] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. I, Springer-Verlag, Berlin, 1983.
- [16] V. Ivrii, Sharp spectral asymptotics for operators with irregular coefficients II. Domains with boundaries and degenerations, *Comm. Partial Differential Equations* 28 (2003) 103–128.
- [17] K. Kawakubo, *The Theory of Transformation Groups*, Clarendon/Oxford Univ. Press, New York, 1991.
- [18] S.Z. Levendorskii, *Asymptotic Distribution of Eigenvalues*, Kluwer Academic Publ., Dordrecht, 1990.
- [19] D. Montgomery, H. Samelson, Transformation groups of spheres, *Ann. of Math.* 44 (1943) 457–470.
- [20] J.P. Ortega, T.S. Ratiu, *Momentum Maps and Hamiltonian Reduction*, *Progr. in Math.*, vol. 222, Birkhäuser Boston, Boston, MA, 2004.
- [21] P. Ramacher, Reduced Weyl asymptotics for pseudodifferential operators on bounded domains I. The finite group case, *J. Funct. Anal.* (2008), doi:10.1016/j.jfa.2008.02.012, in press.
- [22] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness*, Academic Press, New York, 1975.
- [23] M.A. Shubin, *Pseudodifferential Operators and Spectral Theory*, 2nd ed., Springer-Verlag, Berlin, 2001.
- [24] V.N. Tulovsky, M.A. Shubin, On the asymptotic distribution of eigenvalues of pseudodifferential operators in \mathbb{R}^n , *Math. Trans.* 92 (4) (1973) 571–588.
- [25] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung), *Math. Ann.* 71 (1912) 441–479.